

Problem Set #7

Solutions

1. The Legendre polynomials can be obtained from the generating function

$$G(z, h) = (1 - 2zh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z) h^n$$

Use the generating function to find the first six Legendre polynomials ($n = 0$ to $n = 5$).

The expansion of the generating function in powers of h is

$$G(z, h) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{d^{(n)}G}{dh^{(n)}} \right)_{h=0} h^n$$

so we can make the identification,

$$P_n(z) = \frac{1}{n!} \left(\frac{d^{(n)}G}{dh^{(n)}} \right)_{h=0}.$$

We will need the first five derivatives of the generating function:

$$G^{(0)}(z, h) = G = (1 - 2zh + h^2)^{-1/2}$$

$$G^{(1)}(z, h) = (z - h) \cdot (1 - 2zh + h^2)^{-3/2}$$

$$G^{(2)}(z, h) = -(1 - 2zh + h^2)^{-3/2} + 3(z - h)^2 \cdot (1 - 2zh + h^2)^{-5/2}$$

$$G^{(3)}(z, h) = -9(z - h) \cdot (1 - 2zh + h^2)^{-5/2} + 15(z - h)^3 \cdot (1 - 2zh + h^2)^{-7/2}$$

$$G^{(4)}(z, h) = 9(1 - 2zh + h^2)^{-5/2} - 90(z - h)^2 \cdot (1 - 2zh + h^2)^{-7/2} \\ + 105(z - h)^4 \cdot (1 - 2zh + h^2)^{-9/2}$$

$$G^{(5)}(z, h) = 225(z - h) \cdot (1 - 2zh + h^2)^{-7/2} - 1050(z - h)^3 \cdot (1 - 2zh + h^2)^{-9/2} \\ + 945(z - h)^5 \cdot (1 - 2zh + h^2)^{-11/2}$$

Now, evaluate at $h = 0$ to get the Legendre polynomials:

$$P_0(z) = G(z, 0) = 1$$

$$P_1(z) = \frac{1}{1!} G^{(1)}(z, 0) = z$$

$$P_2(z) = \frac{1}{2!} G^{(2)}(z, 0) = \frac{1}{2} (3z^2 - 1)$$

$$P_3(z) = \frac{1}{3!} G^{(3)}(z, 0) = \frac{1}{6} (-9z + 15z^3) = \frac{1}{2} (5z^3 - 3z)$$

$$P_4(z) = \frac{1}{4!} G^{(4)}(z, 0) = \frac{1}{24} (105z^4 - 90z^2 + 9) = \frac{1}{8} (35z^4 - 30z^2 + 3)$$

$$P_5(z) = \frac{1}{5!} G^{(5)}(z, 0) = \frac{1}{120} (945z^5 - 1050z^3 + 225z) = \frac{1}{8} (63z^5 - 70z^3 + 15z)$$

2. RHB Problem 17.9.

The first step is to find the normalized eigenfunctions, i.e. the functions $y_n(x)$ that satisfy the eigenvalue equation $Ly_n = \frac{d^2 y_n}{dx^2} + \kappa y_n = \lambda_n y_n$. Re-writing, we see that the differential equation is the homogeneous simple harmonic oscillator equation:

$$\frac{d^2 y_n}{dx^2} + (\kappa - \lambda_n) y_n = 0.$$

The general solution is

$$y_n(x) = A \sin \sqrt{\kappa - \lambda_n} x + B \cos \sqrt{\kappa - \lambda_n} x$$

The boundary condition $y_n(0) = 0$ implies $B = 0$.

The boundary condition $y_n(\pi) = 0$ implies $\sqrt{\kappa - \lambda_n} = n$ where n is an integer.

Thus $y_n(x) = A \sin nx$.

To normalize, use the definite integral $\int_0^\pi \sin^2 nx \, dx = \frac{\pi}{2}$:

$$\int_0^\pi y_n^*(x) y_n(x) \, dx = A^2 \int_0^\pi \sin^2 nx \, dx = A^2 \frac{\pi}{2} = 1 \Rightarrow A = \left(\frac{2}{\pi}\right)^{1/2}$$

and the normalized eigenfunctions are $y_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin nx$

We will also need the eigenvalues λ_n such that $Ly_n = \frac{d^2 y_n}{dx^2} + \kappa y_n = \lambda_n y_n$.

But, since $\frac{d^2 y_n}{dx^2} = -n^2 y_n$ we have $Ly_n = (\kappa - n^2) y_n \Rightarrow \lambda_n = \kappa - n^2$.

The Green's function is therefore

$$G(x, z) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} \cdot y_n(x) y_n^*(z) = \left(\frac{2}{\pi}\right) \sum_{n=0}^{\infty} \frac{\sin nx \sin nz}{(\kappa - n^2)}$$

The solution to the inhomogeneous equation can now be obtained from the Green's function:

$$y(x) = \int_0^\pi G(x, z) f(z) dz = \left(\frac{2}{\pi}\right) \sum_{n=0}^{\infty} \frac{\sin nx}{(\kappa - n^2)} \left[\int_0^{\pi/2} z \sin nz \, dz + \int_{\pi/2}^\pi (\pi - z) \sin nz \, dz \right]$$

The integrals are

$$\int_0^{\pi/2} z \sin nz \, dz = \frac{1}{n^2} [\sin u - u \cos u]_0^{\pi/2} = \frac{1}{n^2} \left[(-1)^{\frac{n-1}{2}} \right] \quad \text{if } n \text{ is odd,}$$

$$= 0 \text{ if } n \text{ is even.}$$

$$\int_{\pi/2}^{\pi} (\pi - z) \sin nz \, dz = - \int_{\pi/2}^0 u \sin [n(\pi - u)] \, du = \int_0^{\pi/2} u \sin nu \, du = \frac{1}{n^2} (-1)^{\frac{n-1}{2}}$$

$$\text{Thus, } y(x) = \left(\frac{4}{\pi} \right) \sum_{n \text{ odd}} \frac{\sin nx}{n^2 (\kappa - n^2)} (-1)^{\frac{n-1}{2}}.$$

3. RHB Problem 17.10, parts (a) and (c)

(a)

The eigenvalue equation is $\frac{d^2 y_n}{dx^2} = \lambda_n y_n$. The general solution is

$$y_n(x) = A \sin \sqrt{-\lambda_n} x + B \cos \sqrt{-\lambda_n} x$$

The boundary conditions give $B=0$ and $\sqrt{-\lambda_n} = n$ (n integer). The normalization is

$$\int_0^\pi y_n^*(x) y_n(x) dx = A^2 \int_0^\pi \sin^2 nx dx = A^2 \frac{\pi}{2} = 1 \Rightarrow A = \left(\frac{2}{\pi}\right)^{1/2}$$

so that the normalized eigenfunction is

$$y_n(x) = \left(\frac{2}{\pi}\right)^{1/2} \sin nx \text{ with eigenvalue } \lambda_n = -n^2.$$

The Green's function is

$$G(x, z) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} y_n(x) y_n^*(z) = -\left(\frac{2}{\pi}\right) \sum_{n=0}^{\infty} \frac{1}{n^2} \sin nx \cdot \sin nz$$

(c)

$$G(x, z) = \sum_{n=0}^{\infty} a_n \left(\frac{2}{\pi}\right)^{1/2} \sin nx .$$

$$a_n = \left(\frac{2}{\pi}\right)^{1/2} \int_0^z \sin nx \cdot \frac{x(z-\pi)}{\pi} dx + \left(\frac{2}{\pi}\right)^{1/2} \int_z^\pi \sin nx \cdot \frac{z(x-\pi)}{\pi} dx$$

$$a_n = \left(\frac{2}{\pi}\right)^{1/2} \left[\left[\left(\frac{z}{\pi}\right) \int_0^\pi \sin nx \cdot x dx - \int_0^z \sin nx \cdot x dx - \int_z^\pi \sin nx dx \right] \right]$$

The integrals are:

$$\int_0^\pi \sin nx \cdot x dx = -\frac{n\pi \cos n\pi}{n^2}$$

$$\int_0^z \sin nx \cdot x dx = \frac{1}{n^2} \cdot [\sin nz - nz \cos nz]$$

$$\int_z^\pi \sin nx dx = \frac{1}{n} \cdot [\cos n\pi - \cos nz]$$

$$\begin{aligned} a_n &= \left(\frac{2}{\pi}\right)^{1/2} \cdot \frac{1}{n^2} \cdot \left[-\left(\frac{z}{\pi}\right) n \pi \cos n \pi - \sin n z + n z \cos n z + n z \cos n \pi - n z \cos n z \right] \\ &= -\left(\frac{2}{\pi}\right)^{1/2} \cdot \frac{1}{n^2} \cdot \sin n z \end{aligned}$$

$$\text{Thus, } G(x, z) = \sum_{n=0}^{\infty} a_n \left(\frac{2}{\pi}\right)^{1/2} \sin n x = -\left(\frac{2}{\pi}\right) \sum_{n=0}^{\infty} \frac{1}{n^2} \cdot \sin n x \sin n z.$$

4. RHB Problem 17.13.

The given operator is defined by $Ly = x^2 y'' + 2xy' + \frac{1}{4}y$ where $1 \leq x \leq e$.

Make the change of variables $x = \exp t$ $t = \ln x$ and note that $\frac{dt}{dx} = \frac{1}{x} = e^{-t}$.

The derivatives are

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{dy}{dt} e^{-t} \quad \text{and} \quad \frac{d^2 y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dt} e^{-t} \right) \cdot e^{-t} = e^{-2t} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right)$$

The eigenvalue equation $Ly = x^2 y'' + 2xy' + \frac{1}{4}y = \lambda y$ becomes

$$e^{2t} \cdot e^{-2t} \left(\frac{d^2 y}{dt^2} - \frac{dy}{dt} \right) + 2e^t \cdot e^{-t} \frac{dy}{dt} + \left(\frac{1}{4} - \lambda \right) y = 0$$

$$\frac{d^2 y}{dt^2} + \frac{dy}{dt} + \left(\frac{1}{4} - \lambda \right) y = 0$$

This is the equation of motion for a damped harmonic oscillator. The general solution is

$$y(t) = e^{-at} [c \sin bt + d \cos bt]$$

The boundary conditions give: $y(x=1) = y(t=0) = 0 \Rightarrow d = 0$
 $y(x=e) = y(t=1) = 0 \Rightarrow b = n\pi \quad n=1,2,3,\dots$

Substituting $y(t) = ce^{-at} \sin bt$ into the eigenvalue equation and canceling ce^{-at} gives

$$a^2 \sin bt - 2abc \cos bt - b^2 \sin bt - a \sin bt + b \cos bt + \left(\frac{1}{4} - \lambda \right) \sin bt = 0$$

$$\left(a^2 - b^2 - a + \frac{1}{4} - \lambda \right) \sin bt + b(1 - 2a) \cos bt = 0$$

Setting the coefficient of the cosine term equal to zero, we have $a = 1/2$. Then substituting for a in the coefficient of the sine term and setting it equal to zero we get the eigenvalue, $\lambda = -b^2 = -(n\pi)^2$.

The eigenfunctions are thus $y_n(t) = ce^{-t/2} \sin(n\pi t)$ or $y_n(x) = cx^{-1/2} \sin(n\pi \ln x)$.

To normalize, set $\int_1^e c^2 x^{-1} \sin^2(n\pi \ln x) dx = 1$. With $t = \ln x$ and $\frac{dt}{dx} = x^{-1}$, the normalization integral can be written

$$c^2 \int_0^1 \sin^2(n\pi t) dt = \frac{c^2}{n\pi} \int_0^{n\pi} \sin^2 u du = \frac{c^2}{n\pi} \left[\frac{u}{2} + \frac{1}{4} \sin 2u \right]_0^{n\pi} = \frac{c^2}{2} = 1$$

and the normalized eigenfunctions are $y_n(x) = \sqrt{2} x^{-1/2} \sin(n\pi \ln x)$.

Now the Green's function is

$$G(x, z) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} y_n(x) y_n(z) = \sum_{n=0}^{\infty} \frac{-1}{(n\pi)^2} \sqrt{2} x^{-1/2} \sin(n\pi \ln x) \cdot \sqrt{2} z^{-1/2} \sin(n\pi \ln z).$$

The solution to the inhomogeneous equation is

$$y(x) = \int_1^e G(x, z) z^{-1/2} dz = \sum_{n=0}^{\infty} \left[\frac{-\sqrt{2}}{(n\pi)^2} \int_1^e z^{-1} \sin(n\pi \ln z) dz \right] \cdot \left[\sqrt{2} x^{-1/2} \sin(n\pi \ln x) \right] = \sum_{n=0}^{\infty} a_n y_n(x)$$

The coefficients in the expansion are

$$\begin{aligned} a_n &= \frac{-\sqrt{2}}{(n\pi)^2} \int_1^e z^{-1} \sin(n\pi \ln z) dz = \frac{-\sqrt{2}}{(n\pi)^2} \int_0^1 \sin(n\pi t) dt = \frac{-\sqrt{2}}{(n\pi)^2} \cdot \frac{1}{n\pi} [-\cos(n\pi t)]_0^1 \\ &= \frac{-\sqrt{8}}{(n\pi)^3} \quad n \text{ odd} \\ &= 0 \quad n \text{ even} \end{aligned}$$

5. RHB Problem 18.4

Carry through the following procedure to prove the result

$$\int_{-1}^1 P_n(z)P_n(z)dz = \frac{2}{2n+1}.$$

- (a) Square both sides of the generating function definition of the Legendre polynomials given in problem # 1 above.

$$(1-2zh+h^2)^{-1} = \left[\sum_{n=0}^{\infty} P_n(z)h^n \right]^2$$

- (b) Express the squared sum as a new power series in h .

$$\begin{aligned} \left[\sum_{n=0}^{\infty} P_n(z)h^n \right]^2 &= (P_0 + P_1h + P_2h^2 + \dots)(P_0 + P_1h + P_2h^2 + \dots) \\ &= P_0^2 + P_1^2 h^2 + P_2^2 h^4 + \dots + 2P_0P_1h + 2P_0P_2h^2 + \dots \\ &= \sum_{n=0}^{\infty} P_n^2 h^{2n} + \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} P_n P_m h^{m+n} \end{aligned}$$

- (c) Integrate the new power series from $z = -1$ to $z = 1$ and use the orthogonality property of the Legendre polynomials.

The integrals of the cross terms will vanish due to the orthogonality of the Legendre polynomials. Thus,

$$\int_{-1}^1 \left[\sum_{n=0}^{\infty} P_n(z)h^n \right]^2 dz = \int_{-1}^1 P_0^2(z)dz + h^2 \int_{-1}^1 P_1^2(z)dz + \dots = \sum_{n=0}^{\infty} h^{2n} \int_{-1}^1 P_n^2(z)dz$$

- (d) Similarly integrate the square of the generating function and expand the result in powers of h .

$$\begin{aligned} \int_{-1}^1 \frac{dz}{(1-2zh+h^2)} &= -\frac{1}{2h} \ln(1-2zh+h^2) \Big|_{-1}^1 = -\frac{1}{2h} \left[\ln(1-2h+h^2) - \ln(1+2h+h^2) \right] \\ &= \frac{1}{h} \left[\ln(1+h) - \ln(1-h) \right] \end{aligned}$$

Now expand using the Maclaurin series: $\ln(1 \pm x) = \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4 \dots$

$$\begin{aligned} \ln(1+h) - \ln(1-h) &= \left[h - \frac{1}{2}h^2 + \frac{1}{3}h^3 - \frac{1}{4}h^4 + \frac{1}{5}h^5 - \dots \right] - \left[-h - \frac{1}{2}h^2 - \frac{1}{3}h^3 - \frac{1}{4}h^4 - \frac{1}{5}h^5 - \dots \right] \\ &= 2h + \frac{2}{3}h^3 + \frac{2}{5}h^5 + \dots \end{aligned}$$

$$\text{Thus, } \int_{-1}^1 \frac{dz}{(1-2zh+h^2)} = 2 + \frac{2}{3}h^2 + \frac{2}{5}h^4 + \dots = \sum_{n=0}^{\infty} \frac{2}{2n+1} h^{2n}$$

(e) Compare coefficients obtained in steps (c) and (d).

Equating the coefficients of h^{2n} :

$$\int_{-1}^1 P_n^2(z) dz = \int_{-1}^1 P_n(z) P_n(z) dz = \frac{2}{2n+1}$$

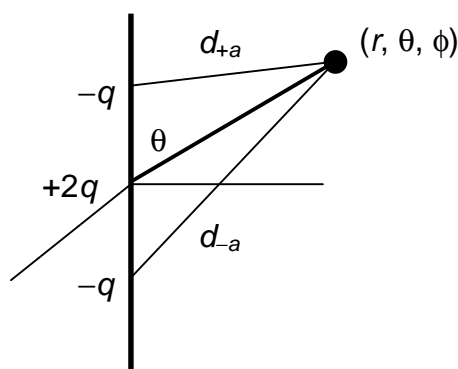
Additional problem for practice -- **REQUIRED FOR PH561 STUDENTS!**

6. RHB Problem 18.6.

Note: I believe that the statement of the problem, at least in my book, is missing an initial minus sign, i.e.

$$\Phi(r, \theta, \phi) = -\frac{2q}{4\pi\epsilon_0 r} \sum_{s=1}^{\infty} \left(\frac{a}{r}\right)^{2s} P_{2s}(\cos \theta).$$

Let d_{+a} , r , and d_{-a} be the distances from the point (r, θ, ϕ) to the charges located on the polar axis at $r = +a$, 0 , and $-a$ respectively:



The distances to the negative charges are $d_{\pm a} = \sqrt{r^2 + a^2 \mp 2ra \cos \theta}$ so that the electrostatic potential is

$$\begin{aligned} \Phi(r, \theta, \phi) &= \frac{q}{4\pi\epsilon_0} \left[\frac{2}{r} - \frac{1}{\sqrt{r^2 + a^2 - 2ra \cos \theta}} - \frac{1}{\sqrt{r^2 + a^2 + 2ra \cos \theta}} \right] \\ &= \frac{2q}{4\pi\epsilon_0 r} \left[1 - \frac{1}{2} \left(1 - 2\left(\frac{r}{a}\right) \cos \theta + \left(\frac{r}{a}\right)^2 \right)^{-1/2} - \frac{1}{2} \left(1 + 2\left(\frac{r}{a}\right) \cos \theta + \left(\frac{r}{a}\right)^2 \right)^{-1/2} \right] \end{aligned}$$

Now, by comparison with the generating function for the Legendre polynomials,

$G(z, h) = (1 - 2zh + h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z) h^n$, we can write

$$\left(1 \mp 2\left(\frac{a}{r}\right) \cos \theta + \left(\frac{a}{r}\right)^2 \right)^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos \theta) \left(\pm \frac{a}{r} \right)^n$$

so that the potential is

$$\begin{aligned}\Phi(r, \theta, \phi) &= \frac{2q}{4\pi\epsilon_0 r} \left[1 - \frac{1}{2} \sum_{n=0}^{\infty} P_n(\cos \theta) \left\{ \left(\frac{a}{r} \right)^n + \left(-\frac{a}{r} \right)^n \right\} \right] \\ &= \frac{2q}{4\pi\epsilon_0 r} \left[1 - \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} P_n(\cos \theta) \left(\frac{a}{r} \right)^n \right] = -\frac{2q}{4\pi\epsilon_0 r} \sum_{s=1}^{\infty} P_s(\cos \theta) \left(\frac{a}{r} \right)^{2s}\end{aligned}$$

where we have used the fact that $P_0(\cos \theta) = 1$.