Problem Set #5

Solutions

1. RHB Problem 16.1.

First, put the given equation into "standard form":

$$y'' - \frac{3z}{1-z^2}y' + \frac{\lambda}{1-z^2}\,y = 0$$

Now, note that $z = 0$ is an ordinary point since $p(z)$ and $q(z)$ are finite and analytic at $z = 0$ and

$$s(z) = z\,p(z) = -\frac{3z^2}{1-z^2} \qquad \Rightarrow \qquad s(0) = 0$$
$$t(z) = z^2\,q(z) = \frac{\lambda z^2}{1-z^2} \qquad \Rightarrow \qquad t(0) = 0$$

Thus, the indicial equation is $\sigma(\sigma-1) = 0$ with roots $\sigma = 1$ and 0 and we can find power series solutions of the form $y_1(z) = \sum_{n=0}^{\infty} a_n z^n$ and $y_2(z) = z\sum_{n=0}^{\infty} b_n z^n$.

For the root $\sigma = 0$, we have

$$\sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1) + s(z)(n+\sigma) + t(z)\right]a_n z^n = 0$$
$$= \sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1) - \frac{3z^2}{1-z^2}(n+\sigma) + \frac{\lambda z^2}{1-z^2}\right]a_n z^n$$
$$= \sum_{n=0}^{\infty}\left[n(n-1) - \frac{3z^2}{1-z^2}n + \frac{\lambda z^2}{1-z^2}\right]a_n z^n = 0$$

Multiply through by $1 - z^2$ and collect terms proportional to z^2:

$$\sum_{n=0}^{\infty}\left[n(n-1)(1-z^2) - 3z^2 n + \lambda z^2\right]a_n z^n$$
$$= \sum_{n=0}^{\infty}\left[z^2\{-n(n-1) - 3n + \lambda\} + n(n-1)\right]a_n z^n$$
$$= \sum_{n=0}^{\infty}\left[z^2\{-n(n-1) - 3n + \lambda\} + n(n-1)\right]a_n z^n$$
$$= \sum_{n=0}^{\infty}\left[-z^2\{n(n+2) - \lambda\} + n(n-1)\right]a_n z^n = 0$$

Now, set the coefficient of z^{n+2} equal to 0:

$$-\{n(n+2)-\lambda\}a_n + (n+2)(n+1)a_{n+2} = 0$$

and the recursion relation for $\sigma = 0$ is $\quad a_{n+2} = a_n \cdot \dfrac{n(n+2)-\lambda}{(n+2)(n+1)}$.

The series will terminate to give a polynomial of order N if $\lambda = N(N+2)$ because then the coefficient of the next term, $a_{N+2} = 0$.

To get a polynomial of order 2, we want $\lambda = 2 \times 4 = 8$. Then we have

$$a_2 = a_0 \cdot \frac{-\lambda}{(2)(1)} = -\frac{8}{2}a_0 = -4a_0 \quad \text{and} \quad U_2(z) = a_0\left(1 - 4z^2\right)$$

For the <u>root $\sigma = 1$</u>, we have

$$\sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1) - \frac{3z^2}{1-z^2}(n+\sigma) + \frac{\lambda z^2}{1-z^2}\right]b_n z^n = 0$$

$$= \sum_{n=0}^{\infty}\left[n(n+1) - \frac{3z^2}{1-z^2}(n+1) + \frac{\lambda z^2}{1-z^2}\right]b_n z^n = 0$$

Again, multiply through by $1 - z^2$ and collect terms proportional to z^2:

$$\sum_{n=0}^{\infty}\left[n(n+1)(1-z^2) - 3z^2(n+1) + \lambda z^2\right]b_n z^n = 0$$

$$= \sum_{n=0}^{\infty}\left[-z^2\{(n+3)(n+1)-\lambda\} + n(n+1)\right]b_n z^n = 0$$

Now, set the coefficient of z^{n+2} equal to 0:

$$-\{(n+3)(n+1)-\lambda\}a_n + (n+2)(n+3)a_{n+2} = 0$$

and the recursion relation for $\sigma = 1$ is $\quad a_{n+2} = a_n \cdot \dfrac{(n+3)(n+1)-\lambda}{(n+2)(n+3)}$..

The series will terminate to give a polynomial of order N + 1 if $\lambda = (N+3)(N+1)$ or a polynomial of order N if $\lambda = N(N+2)$ as before. (Note that if the series terminates after the term in a_2, the polynomial will be of order 3 for $\sigma = 1$.) Thus to get the third order polynomial, we need $\lambda = 3 \times 5 = 15$. Then the coefficient

$$a_2 = \frac{3\cdot 1 - 15}{2\cdot 3}\cdot a_0 = -2 \quad \text{and} \quad U_3(z) = a_0\left(z - 2z^3\right)$$

2. RHB Problem 16.2.

First, put the ODE into the standard form: $y''(z)+p(z)y'(z)+q(z)y=0$:

$$y''+\frac{(1-z)}{2z}y'-\frac{1}{4z}y=0 \text{ where } p(z)=\frac{(1-z)}{2z} \text{ and } q(z)=-\frac{1}{4z}.$$

Now consider the functions $s(z)=zp(z)=\frac{(1-z)}{2}$ *and* $t(z)=z^2q(z)=-\frac{z}{4}$ and note that both **$s(z)$ and $t(z)$ are finite and analytic at the point $z = 0$**. Thus, $z = 0$ is a regular singular point of this ODE.

Since $z = 0$ is a regular singular point, we can assume solutions in the form of Frobenius series: $y(z)=z^\sigma\sum_{n=0}^{\infty}a_nz^n=\sum_{n=0}^{\infty}a_nz^{n+\sigma}$. The derivatives are therefore

$$y'(z)=\sum_{n=0}^{\infty}(n+\sigma)a_nz^{n+\sigma-1} \quad \text{and} \quad y''(z)=\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nz^{n+\sigma-2}.$$

Substitute in the differential equation:

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_nz^{n+\sigma-2}+\left(\frac{1-z}{2z}\right)\cdot\sum_{n=0}^{\infty}(n+\sigma)a_nz^{n+\sigma-1}-\left(\frac{1}{4z}\right)\cdot\sum_{n=0}^{\infty}a_nz^{n+\sigma}=0.$$

Divide through by $z^{\sigma-2}$:

$$\sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1)+\left(\frac{1-z}{2}\right)(n+\sigma)-\frac{z}{4}\right]a_nz^n=0. \qquad \text{(Eq. 1)}$$

(Alternatively, one can start with the "master equation" and substitute $s(z)$ and $t(z)$ to obtain Eq. 1)

Now, let z =0 so that all terms with $n > 0$ vanish. The $n = 0$ term is $\left[\sigma(\sigma-1)+\frac{1}{2}\sigma\right]a_0=0$.

Since $a_0 \neq 0$, we have $\left[\sigma(\sigma-1)+\frac{1}{2}\sigma\right]=\sigma\left(\sigma-\frac{1}{2}\right)=0$.

This is the **indicial equation**. The roots are **σ = ½** and **σ = 0.**

For σ = ½, the Frobenius series becomes = $y_1(z) = z^{1/2}\sum_{n=0}^{\infty} a_n z^n$

$$\sum_{n=0}^{\infty}\left[\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)+\left(\frac{1-z}{2}\right)\left(n+\frac{1}{2}\right)-\frac{z}{4}\right]a_n z^n$$
$$=\sum_{n=0}^{\infty} n\left(n+\frac{1}{2}\right)a_n z^n - \sum_{n=0}^{\infty}\frac{1}{2}(n+1)a_n z^{n+1} = 0$$

Setting the coefficients of z^n to zero: $n\left(n+\frac{1}{2}\right)a_n = \frac{1}{2}na_{n-1} = 0$.

Thus the **recursion relation** is $a_n = \dfrac{\frac{1}{2}n}{n\left(n+\frac{1}{2}\right)}a_{n-1} = \dfrac{1}{2n+1}a_{n-1}$.

The series solution is $y_1(z) = a_0\left[z^{1/2} + \frac{1}{3\cdot 1}z^{3/2} + \frac{1}{5\cdot 3\cdot 1}z^{5/2} + \frac{1}{7\cdot 5\cdot 3\cdot 1}z^{7/2} + \cdots\right]$

The coefficients can be written $\dfrac{2^n n!}{(2n+1)!}$. For example, when n = 3,

$$\frac{2\cdot 2\cdot 2\cdot 3\cdot 2\cdot 1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{6\cdot 4\cdot 2}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = \frac{1}{7\cdot 5\cdot 3\cdot 1}$$

Thus $y_1(z) = a_0\sum_{n=0}^{\infty}\dfrac{2^n n!}{(2n+1)!}z^{n+1/2}$

For the other root, σ = 0, the Frobenius series becomes simply $y_2(z) = \sum_{n=0}^{\infty} b_n z^n$.

Eq. (1) becomes

$$\sum_{n=0}^{\infty}\left[n(n-1)+\left(\frac{1-z}{2}\right)n-\frac{z}{4}\right]b_n z^n = \sum_{n=0}^{\infty}\left[n(n-1)+\frac{n}{2}\right]b_n z^n - \sum_{n=0}^{\infty}\left[\frac{n}{2}+\frac{1}{4}\right]$$
$$=\sum_{n=0}^{\infty} n\left(n-\frac{1}{2}\right)b_n z^n - \sum_{n=0}^{\infty}\frac{1}{2}\left(n+\frac{1}{2}\right)b_n z^{n+1} = 0$$

In order that all coefficients of z^n vanish independently, it is necessary that

$$n\left(n-\frac{1}{2}\right)b_n - \frac{1}{2}\left[(n-1)+\frac{1}{2}\right]b_{n-1} = 0\,.$$

Thus the recurrence relation is $b_n = \frac{1}{2n}b_{n-1}$ and we have

$$b_n = \frac{1}{2n}\cdot\frac{1}{2(n-1)}\cdot\frac{1}{2(n-2)}\cdot\cdots\frac{1}{2\cdot 1}\cdot b_0 = \frac{1}{2^n}\cdot\frac{1}{n!}\cdot b_0$$

The solution is $y_2(z) = \sum_{n=0}^{\infty} b_n z^n = b_0 \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\left(\frac{z}{2}\right)^2 = b_0 \exp(z/2)$.

Show that $y_2(z)$ satisfies the equation $y'' + \frac{1-z}{2z}y' - \frac{1}{4z}y = 0$:

Note that $y_2' = \frac{1}{2}b_0 \exp(z/2) = \frac{1}{2}y_2$ and $y_2'' = \frac{1}{4}b_0 \exp(z/2) = \frac{1}{4}y_2$.

Thus, on substitution in the differential equation we have,

$$\frac{1}{4}y_2 + \frac{1}{2z}\cdot\frac{1}{2}y_2 - \frac{1}{2}\cdot\frac{1}{2}y_2 - \frac{1}{4z}y_2 = 0$$

3. RHB Problem 16.3.

The given O.D.E is $zy'' - 2y' + 9z^5 y = 0$.

First, put into standard form: $y'' + p(z)y' + q(z)y = y'' - \frac{2}{z}y' + 9z^4 y = 0$.

The equation is singular at $z = 0$, but $s(z) = zp(z) = -2$ and $t(z) = z^2 q(z) = 9z^6$ are finite and analytic at this point, so $z = 0$ is a regular singular point.

First solution:

Assume a Frobenius series solution $y_1(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n+\sigma}$.

Differentiating and substituting in the O.D.E,

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-2} + \frac{s(z)}{z}\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} + \frac{t(z)}{z^2}\sum_{n=0}^{\infty}a_n z^{n+\sigma} = 0$$

$$\sum_{n=0}^{\infty}[(n+\sigma)(n+\sigma-1) + s(z)(n+\sigma) + t(z)]a_n z^{n+\sigma-2} = 0$$

Multiplying through by $z^{-\sigma+2}$:

$$\sum_{n=0}^{\infty}[(n+\sigma)(n+\sigma-1) + s(z)(n+\sigma) + t(z)]a_n z^n = 0. \qquad \text{Eq. (1)}$$

At $z = 0$, the only non-vanishing term is the $n = 0$ term,

$$\sigma(\sigma-1) + s(0)\sigma + t(0) = \sigma(\sigma-1) - 2\sigma = \sigma(\sigma-3) = 0.$$

This is the indicial equation with roots $\sigma_1 = 3$ and $\sigma_2 = 0$.

Taking the larger root, $\sigma_1 = 3$, we have the series solution

$$y_1(z) = z^3 \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n z^{n+3}.$$

Substituting this series in to the O.D.E. we get Eq. (1) with $\sigma = 3$:

$$\sum_{n=0}^{\infty}\left[(n+3)(n+2) - 2(n+3) + 9z^6\right]a_n z^n = \sum_{n=0}^{\infty}\left[n(n+3) + 9z^6\right]a_n z^n = 0$$

Now, setting the coefficients of $z^n = 0$ yields the recursion relation:

$$n(n+3)a_n + 9a_{n-6} = 0 \quad \Rightarrow \quad a_n = -\frac{9a_{n-6}}{n(n+3)}.$$

The first few non-zero coefficients are

$$a_0 = 1$$
$$a_6 = -\frac{9a_0}{6\cdot 9} = -\frac{a_0}{3!}$$
$$a_{12} = -\frac{9a_6}{12\cdot 15} = \frac{9a_0}{12\cdot 15\cdot 6} = \frac{a_0}{120} = \frac{a_0}{5!}$$
$$a_{18} = -\frac{9a_{12}}{18\cdot 21} = -\frac{9a_0}{18\cdot 21\cdot 120} = -\frac{a_0}{5040} = -\frac{a_0}{7!}$$

In general, $a_{6m} = \frac{(-1)^m}{(2m+1)!}$ where $m = 0, 1, 2, ...$

The series is

$$y_1(z) = \sum_{n=0}^{\infty} a_n z^{n+3} = a_0 z^3 + a_6 z^9 + a_{12} z^{15} + a_{18} z^{21} + ...$$
$$= a_0\left[z^3 - \frac{(z^3)^3}{3!} + \frac{(z^3)^5}{5!} - \frac{(z^3)^7}{7!} + - ....\right] = a_0 \sin(z^3)$$

Thus, $y_1(z) = a_0 \sin(z^3)$

<u>Second solution</u>:

The linearly independent second solution can be obtained by the Wronskian method:

$$y_2(z) = y_1(z)\int^z \frac{1}{y_1^2(z)} \exp\left[-\int^u p(v)dv\right] du$$

$$\int^u p(v)dv = \int^u \frac{-2}{v} dv = -2\ln u \quad \Rightarrow \quad \exp\left[-\int^u p(v)dv\right] = \exp\left[\ln u^2\right] = u^2$$

Hence,

$$y_2(z) = y_1(z)\cdot\int^{z}\frac{u^2\,du}{\sin^2(u^3)} = y_1(z)\cdot\frac{1}{3}\int^{\sqrt[3]{w}}\frac{dw}{\sin^2 w}$$
$$= y_1(z)\cdot\frac{-1}{3}\cot(w)\big|_{\sqrt[3]{w}} = -\frac{a_0}{3}\cot(z^3)\cdot\sin(z^3)$$

where we have made the substitution $w = u^3$ to simplify the integral. The factor $-\frac{a_0}{3}$ can be replaced by b_0 to give a form parallel to the first solution, i.e. $y_2(z) = b_0\sin(z^3)$.

Linear independence:

Evaluate the Wronskian using Eq. (16.3):

$$W = y_1y_2' - y_2y_1' = a_0\sin(z^3)\cdot(-b_0)\sin(z^3)(3z^2) - b_0\cos(z^3)\cdot a_0\cos(z^3)(3z^2)$$
$$= -3a_0b_0z^2\cdot\left[\cos^2(z^3) + \sin^2(z^3)\right] = -3a_0b_0\,z^2$$

(The Wronskian would have the opposite sign if we had defined y_1 and y_2 in the opposite way.)

The Wronskian $\neq 0$ for $z \neq 0$, establishing the independence of the solutions.

Compare with Eq. (16.4):

$$W(z) = C\exp\left[-\int^{z}p(u)du\right] = C\exp\left[2\int^{z}\frac{du}{u}\right] = C\exp[2\ln z] = C\,z^2$$

This is the same as the previous result if we set the constant $C = -3\,a_0b_0$.

4. (a) Show that z = 1 is a regular singular point of Legendre's equation,

$$(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0\,.$$

Put into "standard form": $y'' - \frac{2z}{(1-z^2)}y' + \frac{\ell(\ell+1)}{(1-z^2)} = 0.$ This equation is clearly singular at $z = 1$. Look at $s(z)$ and $t(z)$:

$$s(z) = (z-1)p(z) = (z-1)\cdot\frac{-2z}{(1-z^2)} = \frac{2z}{(z+1)}$$
$$t(z) = (z-1)^2 q(z) = (z-1)^2\cdot\frac{\ell(\ell+1)}{(1-z^2)} = -\frac{\ell(\ell+1)(z-1)}{z+1}$$

Because $s(z)$ and $t(z)$ are finite and analytic at z = 1, this is a <u>regular singular</u> <u>point</u>.

(b) Show that the indicial equation has a double root s = 0 and obtain the recurrence relation for a series solution corresponding to this root.

To proceed with the problem, it is convenient to change variables: $u = z - 1$. (This moves the singularity to $u = 0$.) The O.D.E. becomes

$$y'' + \frac{2(u+1)}{u(u+2)}y' - \frac{\ell(\ell+1)}{u(u+2)}y = 0\,.$$

Then,

$$p(u) = \frac{2(u+1)}{u(u+2)} \quad\Rightarrow\quad s(u) = up(u) = \frac{2(u+1)}{(u+2)}$$
$$q(u) = -\frac{\ell(\ell+1)}{u(u+2)} \quad\Rightarrow\quad t(u) = u^2 q(u) = -\frac{\ell(\ell+1)}{(u+2)}u\,.$$

Use Eq. 16.16 for the indicial equation:

$$\sigma(\sigma-1) + s(0)\sigma + t(0) = 0$$
$$\sigma(\sigma-1) + 1\cdot\sigma + 0 = \sigma^2 = 0$$

Thus, there is a **double root, $\sigma = 0$**.

Write the solution as a Frobenius series with $\sigma = 0$:

$$y(u) = \sum_{n=0}^{\infty} a_n u^n \text{ which is the same as } y(z) = \sum_{n=0}^{\infty} a_n (z-1)^n\,.$$

The derivatives are $y'' = \sum_{n=0}^{\infty} a_n n(n-1) u^{n-2}$ *and* $y' = \sum_{n=0}^{\infty} a_n n u^{n-1}$.

The recursion relation will be easier to derive if we write the O.D.E. in the form

$$(u+2)y'' + \frac{2(u+1)}{u} y' - \frac{\ell(\ell+1)}{u} = 0$$

Now, substitute the derivatives in the O.D.E,

$$\sum_{n=0}^{\infty} \left[(u+2) n(n-1) u^{n-2} + 2(u+1) n u^{n-2} - \ell(\ell+1) u^{n-1} \right] a_n = 0 .$$

(Writing the O.D.E. in the form above assured that the terms in the [] involve only powers of *u*.)

Simplifying, $\sum_{n=0}^{\infty} \left[2n^2 u^n + \{ n(n+1) - \ell(\ell+1) \} u^{n+1} \right] a_n = 0$.

Setting the coefficient of u^n to zero,

$$\{ n(n+1) - \ell(\ell+1) \}_{n \to n-1} a_{n-1} + 2n^2 a_n = 0$$
$$\{ n(n-1) - \ell(\ell+1) \} a_{n-1} + 2n^2 a_n = 0$$

Solve to get the recurrence relation,

$$a_n = -\frac{\{ n(n-1) - \ell(\ell+1) \}}{2n^2} a_{n-1} \quad \text{or} \quad a_{n+1} = \frac{\{ \ell(\ell+1) - n(n+1) \}}{2(n+1)^2} a_n .$$

(c) Show that for integer ℓ, the solution is a polynomial of degree ℓ.

Note that for integer ℓ, the coefficients will be zero for $n > \ell$. Thus the series will terminate when $n = \ell$ and the solution will be a polynomial of degree ℓ.

<u>Alternate solution without change of variable</u>:

Write the "master equation" with $\sigma = 0$,

$$\sum_{n=0}^{\infty} \left[n(n-1) + \frac{2z}{z+1} \cdot n - \frac{\ell(\ell+1)(z-1)}{z+1} \right] a_n (z-1)^n = 0$$

Multiply through by $(z + 1)$ to clear denominators,

$$\sum_{n=0}^{\infty}\left[n(n-1)(z+1) + 2zn - \ell(\ell+1)(z-1)\right]a_n\,(z-1)^n = 0$$

Identify factors $(z - 1)$ since this is the expansion quantity

$$\sum_{n=0}^{\infty}\left[n(n-1)(z-1+2) + 2(z-1+1)n - \ell(\ell+1)(z-1)\right]a_n\,(z-1)^n = 0$$

$$\sum_{n=0}^{\infty}\left[(z-1)\{n(n-1)+2n-\ell(\ell+1)\} + \{2n(n-1)+2n\}\right]a_n\,(z-1)^n = 0$$

$$\sum_{n=0}^{\infty}\left[\{n(n+1)-\ell(\ell+1)\}a_n\,(z-1)^{n+1} + 2n^2\,a_n\,(z-1)^n\right] = 0$$

Now set the coefficient of $(z - 1)^n = 0$:

$$\{n(n+1) - \ell(\ell+1)\}_{n\to n-1}\,a_{n-1} + 2n^2\,a_n = 0$$

$$\{n(n-1) - \ell(\ell+1)\}a_{n-1} + 2n^2\,a_n = 0$$

Solve to get the recurrence relation:

$$a_n = -\frac{\{n(n-1)-\ell(\ell+1)\}}{2n^2}a_{n-1} \quad \text{or} \quad a_{n+1} = \frac{\{\ell(\ell+1)-n(n+1)\}}{2(n+1)^2}a_n$$

Additional problem for practice -- **REQUIRED FOR PH561 STUDENTS!**

5. RHB Problem 16.14.

Laguerre equation: $zy'' + (1-z)y' + \lambda y = 0$.

Put into "standard form": $y'' + \frac{1-z}{z} y' + \frac{\lambda}{z} y = 0$.

$$p(z) = \frac{1-z}{z} \qquad s(z) = zp(z) = 1-z$$
$$q(z) = \frac{\lambda}{z} \qquad t(z) = \lambda z$$

Indicial equation (16.16): $\sigma(\sigma-1) + s(0)\sigma + t(0) = \sigma(\sigma-1) + \sigma = \sigma^2 = 0$.

There is a double root σ = 0. One solution will be the series $y(z) = \sum_{n=0}^{\infty} a_n z^n$.

Substitution in the "master equation" (16.15) gives

$$\sum_{n=0}^{\infty}[n(n-1) + (1-z)n + \lambda z]a_n z^n = \sum_{n=0}^{\infty}\left[n^2 + (\lambda - n)z\right]a_n z^n = 0 .$$

Setting the coefficient of z^n to zero, $n^2 a_n + [\lambda - (n-1)]a_{n-1} = 0$, yields the recurrence relation,

$$a_n = -\frac{\lambda - (n-1)}{n^2} a_{n-1} \quad or \quad a_{n+1} = -\frac{\lambda - n}{(n+1)^2} a_n$$

If λ = *N* (a non-negative integer), the series will terminate when *n* = *N* and subsequent coefficients will be equal to zero.

The first few coefficients are

$$\begin{aligned}
&a_0\\
&a_1 = -\frac{N-0}{1^2}\cdot a_0\\
&a_2 = -\frac{N-1}{2^2}\cdot(-1)\frac{N-0}{1^2}\\
&a_3 = -\frac{N-2}{3^2}\cdot(-1)\frac{N-1}{2^2}\cdot(-1)\frac{N-0}{1^2}
\end{aligned}$$

A general coefficient will be

$$a_n = (-1)^n \frac{N(N-1)(N-2)(N-3)\cdots(N-n+1)}{(n!)^2} a_0 = (-1)^n \frac{N!}{(n^2)!\,(N-n)!} a_0$$

If the polynomials are normalized so that $L_N(0) = N!$, the expression for the polynomial will be

$$L_N(z) = \sum_{n=0}^{N} (-1)^n \frac{(N!)^2}{(n!)^2 (N-n)!} z^n .$$

For the case $N = 3$,

$$\begin{aligned}
L_3(z) &= (3!)^2 \sum_{n=0}^{3} (-1)^n \frac{z^n}{(n!)^2 (3-n)!} = (3!)^2 \cdot \left[\frac{1}{3!} - \frac{z}{1\cdot 2!} + \frac{z^2}{(2!)^2\cdot 1!} - \frac{z^3}{(3!)^2\cdot 0!}\right]\\
&= 36\cdot\left[\frac{1}{6} - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{36}\right] = 6 - 18z + 9z^2 - z^3
\end{aligned}$$

