Problem Set #1

Solutions

1. Consider two complex numbers $z = 3 + 2i$ and $w = 5 - i$ and evaluate the following quantities:

(a) $z + w = (3 + 2i) + (5 - i) = (3+5) + i\,(2-1) = \mathbf{8 + i}$.

(b) $w - z = (5 - i) - (3 + 2i) = (5 - 3) + i\,(-1 - 2) = \mathbf{2 - 3i}$.

(c) $wz = (5 - i) \times (3 + 2i) = (5\times3 - 2i^2) + i(5\times2 - 3\times1) = (15+2) + i(10-3) = \mathbf{17 + 7i}$.

(d)
$$\frac{z}{w} = \frac{3+2i}{5-i} = \frac{(3+2i)(5+i)}{(5-i)(5+i)} = \frac{(15-2)+i(10+3)}{25+1} = \frac{13}{26} + i\frac{13}{26} = \frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1+i)$$

(e) $z^*w + w^*z = (3-2i)\times(5-i) + (5+i)\times(3+2i)$

$= (15-2) + i(-3-10) + (15-2) + i(10+3) = 13 - 13i + 13 + 13i = \mathbf{26}$.

(f) $z^2 = (3 + 2i) \times (3 + 2i) = (9 - 4) + i(6 + 6) = \mathbf{5 + 12i}$.

(g) $|z|^2 = z^*z = (3 - 2i) \times (3 + 2i) = 9 + 4 = \mathbf{13}$. **Note!** $\boldsymbol{z^2 \neq |z|^2}$.

(h) $|z||w| = \sqrt{3^2 + 2^2} \times \sqrt{5^2 + (-1)^2} = \sqrt{13} \times \sqrt{26} = \sqrt{13 \times 13 \times 2} = 13\sqrt{2}$.

2. The complex impedance of elements in series in an AC circuit is just the sum of the impedances of the individual elements. For a resistor R, a capacitor C, and an inductor L, the individual impedances are: $Z_R = R$, $Z_C = 1/i\omega C$, and $Z_L = i\omega L$ where ω is the AC frequency.

Evaluate the series impedance, $Z = Z_R + Z_C + Z_L$, and show that the imaginary part of the impedance (called the "reactance") vanishes when $\omega^2 = 1/LC$ ("resonance").

$$Z = Z_R + Z_C + Z_L = R + \frac{1}{i\omega C} + i\omega L = R - \frac{i}{\omega C} + i\omega L = R + i\left(\omega L - \frac{1}{\omega C}\right).$$

If $\omega^2 = 1/LC$, $\text{Im}\,Z = \omega L - \frac{1}{\omega C} = \omega \times \frac{1}{\omega^2 C} - \frac{1}{\omega C} = 0$.