Adjoint and Hermitian Operators

In the following, L is a linear operator. The **adjoint** of L is an operator L^+ defined by

$$\int_a^b f^*(x)[Lg(x)]\,\rho(x)dx = \left\{\int_a^b g^*(x)[L^+ f(x)]\,\rho(x)dx\right\}^*$$

These integrals are inner products, which in bra-ket notation would be written,

$$\langle f|Lg\rangle = \langle g|L^+ f\rangle^* .$$

An operator is **self-adjoint** or **Hermitian** if it is equal to its own adjoint, i.e. if

$$\int_a^b f^*(x)[Lg(x)]\rho(x)dx = \left\{\int_a^b g^*(x)[Lf(x)]\rho(x)dx\right\}^*$$ or, equivalently, $\langle f|Lg\rangle = \langle g|Lf\rangle^*$.

Important properties of Hermitian operators:

If there exists a set of (eigen) functions $y_i(x)$ such that $Ly_i(x) = \lambda_l\,\rho(x)\, y_i(x)$ where the λ_i are constants (eigenvalues), then

(1) **the eigenvalues are real if L is a Hermitian operator,**

(2) **the eigenfunctions belonging to different eigenvalues are orthogonal if L is a Hermitian operator.**

(3) **if two or more eigenfunctions have the same eigenvalue (degeneracy), a set of orthogonal eigenfunctions can be constructed by appropriate linear combinations of the degenerate eigenfunctions.**

Note: If the eigenfunctions are normalized so that $\langle \phi_i|\phi_i\rangle = \int_a^b \phi_i^*(x)\phi_i(x)\rho(x)dx = 1$, then the eigenfunctions are orthonormal (orthogonal and normalized). That is,

$$\langle \phi_i|\phi_j\rangle = \int_a^b \phi_i^*(x)\phi_j(x)\rho(x)dx = \delta_{ij} .$$

