

Series Solutions of Homogeneous, Linear, 2nd Order ODE

“Standard form” of a homogeneous, linear, 2nd order ODE:

$$y'' + p(z)y' + q(z)y = 0$$

The first step is always to put the given equation into this “standard form.” Then the technique for obtaining two independent solutions depends on the existence and nature of any singularities of the ODE at the point $z = z_0$ about which the series is to be expanded. By a change of variable, the ODE can always be written so that $z_0 = 0$ and this is assumed below.

Case I: $z = 0$ is an ordinary point:

- (1) Assume a power series solution about $z = 0$:

$$y(z) = \sum_{n=0}^{\infty} a_n z^n. \quad \text{Eq. (1)}$$

- (2) Substitute Eq. (1) and its derivatives into the ODE and require that the coefficients of each power of z sum to zero to get a **recurrence relation**. The recurrence relation expresses a coefficient a_n in terms of the coefficients a_r where $r \leq n - 1$.

- (3) Sometimes the recurrence relation will have two solutions which can be used to construct series for two independent solutions to the ODE, $y_1(z)$ and $y_2(z)$. Other times, independent solutions can be obtained by choosing differing starting points for the series, e.g. $a_0 = 1, a_1 = 0$ or $a_0 = 0, a_1 = 1$. A systematic approach is to assume **Frobenius power series** solutions

$$y(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n \quad \text{Eq. (2)}$$

with $\sigma = 0$ and $\sigma = 1$ instead of using only Eq. (1).

- (4) Try to relate the power series to the closed form of an elementary function (not always possible).
- (5) Verify your solutions by direct substitution into the ODE.
- (6) The general solution will be $y(z) = c_1 y_1(z) + c_2 y_2(z)$ where c_1 and c_2 are constants that may be determined by initial or boundary conditions.

Case II: $z = 0$ is a regular singular point:

- (1) Assume a **Frobenius power series** solution, Eq. (2), where $a_0 \neq 0$.
- (2) Substitute Eq. (2) and its derivatives into the ODE and set $z = 0$ to obtain the **indicial equation**

$$\sigma(\sigma - 1) + s(0)\sigma + t(0) = 0 \quad \text{where} \quad s(z) = zp(z) \text{ and } t(z) = z^2q(z) \quad \text{Eq. (3)}$$

- (3) Solve the indicial equation to obtain its two roots σ_1 and σ_2 . There are two possible outcomes: σ_1 and σ_2 differ by an integer, or they do not.

Case IIa: σ_1 and σ_2 do not differ by an integer:

In this case, two independent solutions can be obtained by assuming Frobenius power series solutions using the two roots of the indicial equation.

$$y_1(z) = z^{\sigma_1} \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad y_2(z) = z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n$$

- (1) For each value of σ , substitute the corresponding Frobenius series and its derivatives into the ODE to obtain a recurrence relation (procedure similar to Case I, ordinary point).
- (2) Use the recurrence relations to construct each series.
- (3) Express the series, if possible, in terms of elementary functions.
- (4) Verify your solutions by direct substitution into the ODE.
- (5) The general solution will be $y(z) = c_1 y_1(z) + c_2 y_2(z)$ where c_1 and c_2 are constants that may be determined by initial or boundary conditions.

Case IIb: σ_1 and σ_2 differ by an integer:

- (1) Take the larger of the two roots (or the one with the larger real part if the root is complex) and construct a Frobenius power series solution for $y_1(z)$ as in Case IIa above.
- (2) Obtain a second linearly independent solution $y_2(z)$ using the **Wronskian method** or the **derivative method**.

Wronskian method:

$$y_2(z) = y_1(z) \int^z \frac{1}{y_1^2(u)} \exp\left\{-\int^u p(v)dv\right\} du$$

This method is practical if the integrals can be evaluated. This will depend on the specific ODE whose solution is sought.

Derivative method:

First, obtain the recurrence relation and the coefficients $a_n(\sigma)$ in terms of a variable σ . Then, $y(z, \sigma) = z^\sigma \sum_{n=0}^{\infty} a_n(\sigma) z^n$.

Case 1: integer difference = 0 ($\sigma_1 = \sigma_2$).

In the case of equal roots, a second solution will be given by the derivative

$$y_2(z) = \left[\frac{\partial}{\partial \sigma} y(z, \sigma) \right]_{\sigma=\sigma_2}$$

Case 2: integer difference $\neq 0$.

If the roots differ by a non-zero integer, a second solution will be given by

$$y_2(z) = \left[\frac{\partial}{\partial \sigma} \{(\sigma - \sigma_2)y(z, \sigma)\} \right]_{\sigma=\sigma_2}$$

- (3) Verify your first and second solutions by direct substitution of $y_1(z)$ and $y_2(z)$ into the ODE.
- (4) The general solution will be $y(z) = c_1 y_1(z) + c_2 y_2(z)$ where c_1 and c_2 are constants that may be determined by initial or boundary conditions.

Decision Tree for Series Solution of Homogeneous, Linear, 2nd Order ODE

