Vector Operators and Identities

Differential Operators in Cartesian coordinates:

"del" operator $\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}$

Gradient of a scalar field: $\text{grad}\,\phi = \vec{\nabla}\phi(x,y,z) = \hat{\mathbf{i}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial \phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial \phi}{\partial z}$

Divergence of a vector field: $\text{div}\,\vec{\mathbf{a}} = \vec{\nabla}\cdot\vec{\mathbf{a}} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$

Curl of a vector field:

$$\text{curl}\,\vec{\mathbf{a}} = \vec{\nabla}\times\vec{\mathbf{a}} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\hat{\mathbf{k}}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

Operations on sums and products (after RHB Table 10.1):

$$\vec{\nabla}(\phi + \psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$$

$$\vec{\nabla}\cdot(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \vec{\nabla}\cdot\vec{\mathbf{a}} + \vec{\nabla}\cdot\vec{\mathbf{b}}$$

$$\vec{\nabla}\times(\vec{\mathbf{a}} + \vec{\mathbf{b}}) = \vec{\nabla}\times\vec{\mathbf{a}} + \vec{\nabla}\times\vec{\mathbf{b}}$$

$$\vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$$

$$\vec{\nabla}(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}) = \vec{\mathbf{a}}\times(\vec{\nabla}\times\vec{\mathbf{b}}) + \vec{\mathbf{b}}\times(\vec{\nabla}\times\vec{\mathbf{a}}) + (\vec{\mathbf{a}}\cdot\vec{\nabla})\vec{\mathbf{b}} + (\vec{\mathbf{b}}\cdot\vec{\nabla})\vec{\mathbf{a}}$$

$$\vec{\nabla}\cdot(\phi\vec{\mathbf{a}}) = \phi\vec{\nabla}\cdot\vec{\mathbf{a}} + \vec{\mathbf{a}}\cdot\vec{\nabla}\phi$$

$$\vec{\nabla}\cdot(\vec{\mathbf{a}}\times\vec{\mathbf{b}}) = \vec{\mathbf{b}}\cdot(\vec{\nabla}\times\vec{\mathbf{a}}) - \vec{\mathbf{a}}\cdot(\vec{\nabla}\times\vec{\mathbf{b}})$$

$$\vec{\nabla}\times(\phi\vec{\mathbf{a}}) = \vec{\nabla}\phi\times\vec{\mathbf{a}} + \phi\vec{\nabla}\times\vec{\mathbf{a}}$$

$$\vec{\nabla}\times(\vec{\mathbf{a}}\times\vec{\mathbf{b}}) = \vec{\mathbf{a}}(\vec{\nabla}\cdot\vec{\mathbf{b}}) - \vec{\mathbf{b}}(\vec{\nabla}\cdot\vec{\mathbf{a}}) + (\vec{\mathbf{b}}\cdot\vec{\nabla})\vec{\mathbf{a}} - (\vec{\mathbf{a}}\cdot\vec{\nabla})\vec{\mathbf{b}}$$

Note:

$$\vec{\mathbf{a}}\cdot\vec{\nabla} = a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}$$

$$\left(\vec{\mathbf{a}}\cdot\vec{\nabla}\right)\vec{\mathbf{b}} = \left(a_x\frac{\partial b_x}{\partial x} + a_y\frac{\partial b_x}{\partial y} + a_z\frac{\partial b_x}{\partial z}\right)\hat{\mathbf{i}} + \left(a_x\frac{\partial b_y}{\partial x} + a_y\frac{\partial b_y}{\partial y} + a_z\frac{\partial b_y}{\partial z}\right)\hat{\mathbf{j}} + \left(a_x\frac{\partial b_z}{\partial x} + a_y\frac{\partial b_z}{\partial y} + a_z\frac{\partial b_z}{\partial z}\right)\hat{\mathbf{k}}$$

Combinations of grad, div and curl:

$$\text{curl grad}\,\phi = \vec{\nabla}\times\vec{\nabla}\phi = \mathbf{0}$$

$$\text{div curl}\,\vec{\mathbf{a}} = \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{\mathbf{a}}\right) = 0$$

$$\text{div grad}\,\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\text{grad div}\,\vec{\mathbf{a}} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{a}}\right) = \left(\frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_y}{\partial x\partial y} + \frac{\partial^2 a_z}{\partial x\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial^2 a_x}{\partial y\partial x} + \frac{\partial^2 a_y}{\partial y^2} + \frac{\partial^2 a_z}{\partial y\partial z}\right)\hat{\mathbf{j}} + \left(\frac{\partial^2 a_x}{\partial z\partial x} + \frac{\partial^2 a_y}{\partial z\partial y} + \frac{\partial^2 a_z}{\partial z^2}\right)\hat{\mathbf{k}}$$

$$\text{curl curl}\,\vec{\mathbf{a}} = \vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{a}}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{a}}\right) - \nabla^2\vec{\mathbf{a}}$$

$$\left[\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathbf{a}}\right)\right]_i = \left[\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathbf{a}}\right)\right]_i - \nabla^2 a_i \quad \text{(Cartesian coordinates)}$$

