

## Analytic Functions – Cauchy-Riemann Relations

$z = x + iy$  is a complex number (a point  $x, y$  in the complex plane).

The function  $f(z) = u(x, y) + i v(x, y)$  is **differentiable** if the limit

$$f'(z) = \lim_{\Delta z \rightarrow 0} \left[ \frac{f(z + \Delta z) - f(z)}{\Delta z} \right]$$

exists and is independent of the direction that  $\Delta z \rightarrow 0$  in the complex plane.

The function  $f(z)$  is **analytic** in a domain  $R$  in the complex plane if it is **differentiable** and also **single-valued** within  $R$ .

For  $f(z)$  to be differentiable, it is necessary that  $u(x, y)$  and  $v(x, y)$  satisfy the **Cauchy - Riemann relations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

A sufficient condition for  $f(z)$  to be analytic in  $R$  is that the four partial derivatives satisfy the Cauchy – Riemann relations and are continuous.

If  $u(x, y)$  and  $v(x, y)$  satisfy the Cauchy - Riemann relations, the following is true:

- $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Thus,  $u(x, y)$  and  $v(x, y)$  satisfy the two-dimensional Laplace equation.

- $\bar{\nabla} u \cdot \bar{\nabla} v = 0$

Thus, contours of constant  $u$  and  $v$  in the complex plane cross at right-angles.