Determinants

Every square matrix has a **determinant,** det **A** = |**A**| which is a <u>scalar</u> quantity.

<u>Example (N = 3)</u>:

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

Determinants can be evaluated using **minors** and **cofactors**.

Minor

Every element A_{ij} of the determinant (or matrix) has an associated **minor** M_{ij}. The minor has the value of the determinant formed by elimination of row I and column j.

<u>Example (N = 3):</u> For element A_{23} of the previous example, the minor M_{23} is the determinant $\begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}$.

Cofactor

Every element A_{ij} of the determinant (or matrix) also has an associated **cofactor** C_{ij}. The cofactor C_{ij} is equal to the corresponding minor M_{ij} multiplied by $(-1)^{I+j}$.

<u>Example (N = 3):</u> For element A_{23},

$$\text{cofactor } C_{23} = (-1)^{2+3} \times \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} = - \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix}.$$

The **value of a determinant** is the sum of the products of elements of any row or column times their respective cofactors.

<u>Example (N = 3)</u>: Use elements of the first row (most common choice).

$$\begin{aligned} |\mathbf{A}| &= A_{11}C_{11} + A_{12}C_{12} + A_{13}C_{13} \\ &= A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} \end{aligned}$$

To evaluate 2×2 determinants: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Thus, $\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = A_{22}A_{33} - A_{23}A_{32}$ etc.

Identities and Properties of Determinants

Vector (cross) product in Cartesian 3-space: $\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

$$\vec{a} \cdot \left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\left|\mathbf{A}^{\mathrm{T}}\right| = |\mathbf{A}|$$

$$\left|\mathbf{A}^{+}\right| = \left|\left(\mathbf{A}^{*}\right)^{\mathrm{T}}\right| = \left|\mathbf{A}^{*}\right| = |\mathbf{A}|^{*}$$

Interchange any two rows or columns: $|\mathbf{A}| \rightarrow -|\mathbf{A}|$

$$|\lambda \mathbf{A}| = \lambda^{N} |\mathbf{A}|$$

$|\mathbf{AB}| = |A||B| = |\mathbf{BA}|$ (**A** and **B** are both $N \times N$ matrices)

Any two rows or columns identical or multiples of one another: $|\mathbf{A}| = 0$

All elements of one row or column = 0: $|\mathbf{A}| = 0$

Add to elements of one row (or column) a fixed multiple of elements of another row (or column): determinant is unchanged.

Inverse Matrices

A square matrix with $|\mathbf{A}| = 0$ is **singular**.

If **A** is not singular, can define **inverse matrix** $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

In terms of cofactors: $\left(\mathbf{A}^{-1}\right)_{ij} = \dfrac{C_{ji}}{|\mathbf{A}|} = \dfrac{C_{ij}^{\mathrm{T}}}{|\mathbf{A}|}$.

Properties of the Inverse Matrix

$$\left(\mathbf{A}^{-1}\right)^{-1} = \mathbf{A}$$

$$\left(\mathbf{A}^{\mathrm{T}}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{\mathrm{T}}$$

$$\left(\mathbf{A}^{+}\right)^{-1} = \left(\mathbf{A}^{-1}\right)^{+}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$(\mathbf{AB}\cdots\mathbf{G})^{-1} = \mathbf{G}^{-1}\cdots\mathbf{B}^{-1}\mathbf{A}^{-1}$$

