Operators and Matrices

Linear Operators

A **linear operator** $\mathcal{A}$ associates with every vector **x** a new vector **y**: $\mathbf{y} = \mathcal{A}\,\mathbf{x}$.

Some properties of linear operators:

$$\mathcal{A}(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda\mathcal{A}\,\mathbf{a} + \mu\mathcal{A}\,\mathbf{b}$$

$$(\mathcal{A} + \mathcal{B})\mathbf{x} = \mathcal{A}\mathbf{x} + \mathcal{B}\mathbf{x}$$

$$(\lambda\mathcal{A})\mathbf{x} = \lambda(\mathcal{A}\mathbf{x})$$

$$(\mathcal{A}\mathcal{B})\mathbf{x} = \mathcal{A}(\mathcal{B}\mathbf{x})$$

Note: In general $\mathcal{A}\mathcal{B}\mathbf{x} \neq \mathcal{B}\mathcal{A}\mathbf{x}$ **!!** (operators do not always commute)

Null operator $\mathcal{O}$: $\mathcal{O}\mathbf{x} = \mathbf{0}$

Identity operator $\mathcal{I}$ **:** $\mathcal{I}\,\mathbf{x} = \mathbf{x}$

Inverse operator $\mathcal{A}^{-1}$: $\mathcal{A}^{-1}\mathcal{A} = \mathcal{I}$ (If inverse does not exist, operator is **singular**.)

Matrix Elements

In a particular orthonormal basis $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \ldots\ldots, \hat{\mathbf{e}}_N)$, $\mathcal{A}\,\hat{e}_j = \sum_{i=1}^{N} A_{ij}\hat{e}_i$ where the A_{ij} are the **matrix elements** of the operator $\mathcal{A}$ in the given basis.

If $\mathbf{y} = \mathcal{A}\,\mathbf{x}$**,** then in component form, $y_i = \sum_{j=1}^{N} A_{ij}x_j$ <u>with respect to the given basis</u>.

Matrices

In general, an operator $\mathcal{A}$ can transform a vector **x** in an *N*-dimensional space into a vector **y** in an *M*-dimensional space with basis $\left(\hat{\mathbf{f}}_1, \hat{\mathbf{f}}_2, \hat{\mathbf{f}}_3, \ldots\ldots, \hat{\mathbf{f}}_M\right)$:

$$\mathcal{A}\,\hat{e}_j = \sum_{i=1}^{M} A_{ij}\hat{\mathbf{f}}_i \quad \text{where } j = 1, 2, 3, \ldots\ldots, N.$$

The matrix elements can be arranged in an $M \times N$ array (**matrix**):

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdot & \cdot & \cdot & A_{1N} \\ A_{21} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{M1} & \cdot & \cdot & \cdot & \cdot & A_{MN} \end{pmatrix}$$

Frequently both vector spaces have the same dimension, *N*, yielding an $N \times N$ **square matrix** for an operator $\mathcal{A}$.

A vector **x** in an *N*-dimensional space can be expressed as a column matrix:

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{pmatrix}$$

Matrix Algebra

Addition of two matrices: $(\mathbf{A} + \mathbf{B})_{ij} = A_{ij} + B_{ij}$
(Matrices A and B must have the same dimensions, $M \times N$.)

Multiplication by a scalar: $(\lambda\mathbf{A})_{ij} = \lambda A_{ij}$

Multiplication of two matrices: $\mathbf{C} = \mathbf{AB}$ $\qquad C_{ij} = \sum_{k=1}^{N} A_{ik}B_{kj}$
(Number of columns, *N*, of **A** must equal the number of rows of **B**.)

Example: $\mathbf{y} = \mathcal{A}\,\mathbf{x}$ (**y** in 4-D space, **x** in 6-D space**)**

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

$y_1 = A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + A_{14}x_4 + A_{15}x_5 + A_{16}x_6$

$y_2 = A_{21}x_1 + A_{22}x_2 + A_{23}x_3 + A_{24}x_4 + A_{25}x_5 + A_{26}x_6$

etc.

<u>Example</u>: $\mathbf{C} = \mathbf{AB}$ (**A** is 2×3 matrix, **B** is 3×2 matrix.)

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{pmatrix}$$

$C_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31}$

$C_{21} = A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31}$

$C_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}$

$C_{22} = A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32}$

Null matrix has all elements equal to zero.

Identity matrix (must be square) has “diagonal elements” = 1, all others = 0:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Transpose $\mathbf{A}^{\mathrm{T}}$ of a matrix has interchanged rows and columns of **A**: $A_{ij}^{\mathrm{T}} = A_{ji}$

Example: $$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \end{pmatrix}$$

Complex conjugate of a matrix **A***: $(\mathbf{A}^*)_{ij} = (\mathbf{A}_{ij})^*$

Hermetian conjugate (**adjoint**) $\mathbf{A}^\dagger$ is the transpose of the complex conjugate:

$\mathbf{A}^\dagger = (\mathbf{A}^*)^{\mathrm{T}} = (A^{\mathrm{T}})^*$ (If **A** is real, $\mathbf{A}^\dagger = A^{\mathrm{T}}$)

The **trace** of a square matrix is the sum of the diagonal elements:

$$\mathrm{Tr}\,\mathbf{A} = A_{11} + A_{22} + A_{33} + \ldots\ldots. + A_{NN}$$

Inner product of vectors expressed as matrices:

$$\langle \mathbf{a} | \mathbf{b} \rangle \Rightarrow \mathbf{a}^{+}\mathbf{b} = \begin{pmatrix} \mathbf{a}_1^* & \mathbf{a}_2^* & \mathbf{a}_3^* & \cdot & \cdot & \cdot & \mathbf{a}_N^* \end{pmatrix} \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b}_N \end{pmatrix} = \sum_{i=1}^{N} a_i^* b_i \quad \text{(orthonormal basis only)}$$

