Vectors and Vector Spaces

A **vector** in an abstract *N*-dimensional space can be represented by a set of numbers (components): $\mathbf{a} = (a_1, a_2, a_3, \ldots\ldots, a_N)$

Example: in Cartesian 3-dimensional space: $\vec{a} = \left(a_x,\ a_y,\ a_z\right) = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ where the $\hat{i}, \hat{j}, \hat{k}$ are orthogonal vectors of unit length in the *x*-, *y*-, and *z*-directions, respectively.

Vector Space

A set of vectors **a, b, c, d**, ……. forms a **vector space** if

(i) they obey commutative and associative rules of addition:

$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ is a vector in the same space
$(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$

(ii) they can be multiplied by scalars to produce a new vector in the space:

$\lambda\mathbf{a}$ is a new vector in the space
$\lambda(\mathbf{a} + \mathbf{b}) = \lambda\mathbf{a} + \lambda\mathbf{b}$
$(\lambda + \mu)\mathbf{a} = \lambda\mathbf{a} + \mu\mathbf{a}$
$\lambda(\mu\mathbf{a}) = (\lambda\mu)\mathbf{a}$

(iii) there exists a null vector **0** such that $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all **a**

(iv) multiplication by unity leaves any vector unchanged, $\mathbf{1} \times \mathbf{a} = \mathbf{a}$

(v) all vectors have a negative –**a** such that $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$.

Linear independence

A member of a set of **linearly independent** vectors cannot be expressed as a linear sum of other vectors in the set. For example, in the set of Cartesian unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$, the unit vector $\hat{\mathbf{i}}$ cannot be expressed as a sum $\hat{\mathbf{i}} = c_1\hat{\mathbf{j}} + c_2\hat{\mathbf{k}} =$ for any coefficients c_1 and c_2.

Basis Vectors

A set of linearly independent vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \cdots, \mathbf{e}_N$ forms a **basis** if any other vector in the space can be expressed as a linear sum $\mathbf{a} = \sum_{i=1}^{N} a_i \mathbf{e}_i$ where the a_i are scalar coefficients (components).

Note: the coefficients relate to a particular basis. In another basis set $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3, \cdots, \mathbf{e}'_N$,
$\mathbf{a} = \sum_{i=1}^{N} a'_i \mathbf{e}'_i$ where the $a'_i \neq a_i$ for at least some *i*.

Inner Product

Two vectors **a** and **b** in a vector space can form an **inner product** $\langle \mathbf{a} | \mathbf{b} \rangle$, which is a scalar function.

If the vectors are complex (have complex components), then $\langle \mathbf{a} | \mathbf{b} \rangle = \langle \mathbf{b} | \mathbf{a} \rangle^*$.

Also, $\langle \lambda \mathbf{a} + \mu \mathbf{b} | \mathbf{c} \rangle = \lambda^* \langle \mathbf{a} | \mathbf{c} \rangle + \mu^* \langle b | \mathbf{c} \rangle$ and $\langle \lambda \mathbf{a} | \mu \mathbf{b} \rangle = \lambda^* \mu \langle \mathbf{a} | b \rangle$.

If **a** and **b** are **orthogonal** to one another, $\langle \mathbf{a} | \mathbf{b} \rangle = 0$.

The **norm** of a vector **a** is $\|\mathbf{a}\| = \langle \mathbf{a} | \mathbf{a} \rangle^{1/2}$.

If the vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \ \ , \hat{\mathbf{e}}_N$ of a basis set are orthogonal and have norm = 1, i.e. if $\langle \hat{\mathbf{e}}_i | \hat{\mathbf{e}}_j \rangle = \delta_{ij}$, the set is **orthonormal** ($\delta_{ij} = 0$ if $I \neq j$, $\delta_{ij} = 1$ if $I = j$).

Projection of vector **a** on to basis vector $\hat{\mathbf{e}}_j$: $\langle \hat{\mathbf{e}}_j | \mathbf{a} \rangle = a_j$ (orthonormal basis).

If $\mathbf{a} = \sum_{i=1}^{N} a_i \hat{\mathbf{e}}_i$ and $\mathbf{b} = \sum_{i=1}^{N} b_i \hat{\mathbf{e}}_i$, then $\langle \mathbf{a} | \mathbf{b} \rangle = \sum_{i=1}^{N} a_i^* b_i$ (orthonormal basis).

In ordinary 3-dimensional space, the inner product is the **dot product**: $\langle \mathbf{a} | \mathbf{b} \rangle = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = |\vec{\mathbf{a}}||\vec{\mathbf{b}}| \cos \theta_{ab}$.

Inequalities involving inner products and norms

(i) Schwartz's inequality: $\left|\langle \mathbf{a} | \mathbf{b} \rangle\right| \le \|\mathbf{a}\| \|\mathbf{b}\|$, equal if $\mathbf{a} = \lambda \mathbf{b}$.

(ii) Triangle inequality: $\| \mathbf{a} + \mathbf{b} \| \le \|\mathbf{a}\| + \|\mathbf{b}\|$.

(iii) Bessel's inequality: $\langle \mathbf{a} | \mathbf{b} \rangle \ge \sum_{i=1}^{M} \left| a_i \right|^2$, equal if $M = N$ (dimension of space).

(iv) Parallelogram <u>equality</u>: $\| \mathbf{a} + \mathbf{b} \|^2 + \| \mathbf{a} - \mathbf{b} \|^2 = 2\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 \right)$.