Problem Set #7

Solutions

1. Obtain the Laurent series and identify the residue for the indicated poles in each of the following functions:

(a) $f(z) = \dfrac{e^{2z}}{z^3}$ at $z = 0$.

$$f(z) = \frac{e^{2z}}{z^3} \equiv \frac{g(z)}{z^3}$$

Expand $g(z)$ about $z = 0$:

$$g(z) = 1 + (2z) + \frac{1}{2!}(2z)^2 + \frac{1}{3!}(2z)^3 + \frac{1}{4!}(2z)^4 + \cdots$$
$$= 1 + 2z + 2z^2 + \frac{4}{3}z^3 + \frac{2}{3}z^2 + \cdots$$

Then the Laurent series for $f(z)$ is

$$f(z) = \frac{g(z)}{z^3} = \frac{1}{z^3} + 2\frac{1}{z^2} + 2\frac{1}{z} + \frac{4}{3} + \frac{2}{3}z + \cdots$$

and the residue for the pole of order 3 at $z = 0$ is $a_{-1} = 2$.

(b) $f(z) = \dfrac{1}{z^2(z-1)^4}$ at $z = 0$, $z = 1$.

For the double pole at $z = 0$, let $g(z) = (z - 1)^{-4} = (1 - z)^{-4}$.

Expand:
$$g(z) = (1 - z)^{-4} = 1 + 4z + \frac{(4)(5)}{2!}z^2 + \frac{(4)(5)(6)}{3!}z^3 + \cdots$$
$$= 1 + 4z + 10z^2 + 20z^3 + \cdots$$

and the Laurent series is

$$f(z) = \frac{g(z)}{z^2} = \frac{1}{z^2} + 4\frac{1}{z} + 10 + 20z + \cdots$$

The residue for the pole at $z = 0$ is $a_{-1} = 4$.

For the pole of order 4 at $z = 1$, it is convenient to change variable: $\xi = z - 1$.

Then, $f(\xi) = \dfrac{1}{(\xi+1)^2\xi^4}$ and we now have a pole of order 4 at $\xi = 0$ with

$$g(\xi) = (1+\xi)^{-2} = 1 - 2\xi + 3\xi^2 - 4\xi^3 + 5\xi^4 - 6\xi^5 + \cdots.$$

(One can also retain z as the variable and expand in powers of $(z - 1)$.)

The Laurent series is

$$f(\xi) = \frac{g(\xi)}{\xi^4} = \frac{1}{\xi^4} - \frac{2}{\xi^3} + \frac{3}{\xi^2} - \frac{4}{\xi} + 5 - 6\xi + \cdots$$

and the residue for the pole at $\xi = 0$ ($z = 1$) is $a_{-1} = -4$.