Problem Set #6

Solutions

1. Evaluate $\oint_C \frac{e^z}{z}dz$ where the contour C is the circle $|z| = 1$.

Use Cauchy's integral formula: $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}dz$.

In this case, $f(z) = e^z$ and $z_0 = 0 \;\Rightarrow\; f(z_0) = f(0) = 1$

Thus, $1 = \frac{1}{2\pi i}\oint_C \frac{e^z}{z}dz \qquad \Rightarrow \qquad \oint_C \frac{e^z}{z}dz = 2\pi i$

2. Evaluate $\oint_C \frac{z^2 + 1}{z^2 - 1}dz$ where the contour C is the circle $|z| = 2$.

First, put the integrand in a form so that Cauchy's integral formula can be applied:

$$\frac{z^2+1}{z^2-1} = \frac{z^2+1}{(z+1)(z-1)} = \frac{z}{z-1} - \frac{1}{z+1}.$$

Then $\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z}{z-1}dz - \oint_C \frac{1}{z+1}dz$.

To apply Cauchy's formula to the first integral, let $f(z) = z$ and $z_0 = 1$:

$$f(z_0) = 1 = \frac{1}{2\pi i}\oint_C \frac{z}{z-1}dz \qquad \Rightarrow \qquad \oint_C \frac{z}{z-1}dz = 2\pi i .$$

For the second integral, $f(z) = 1$ and $z_0 = -1$:

$$f(z_0) = 1 = \frac{1}{2\pi i}\oint_C \frac{1}{z+1}dz \qquad \Rightarrow \qquad \oint_C \frac{1}{z+1}dz = 2\pi i$$

Then, $\oint_C \frac{z^2+1}{z^2-1}dz = \oint_C \frac{z}{z-1}dz - \oint_C \frac{1}{z+1}dz = 2\pi i - 2\pi i = 0$

3. RHB Problem 24.10

First show that *f*(*z*) is analytic. Use the Cauchy-Riemann relations.

$$f(z)=e^{iaz^2} = e^{ia\left(x^2-y^2+i2xy\right)} = 2e^{-2axy}\left[\cos\left(ax^2-ay^2\right)+i\sin\left(ax^2-ay^2\right)\right]$$
$$= u(x,y) + i\,v(x,y)$$
$$u(x,y) = 2e^{-2axy}\cos\left(ax^2-ay^2\right) \qquad v(x,y) = 2e^{-2axy}\sin\left(ax^2-ay^2\right)$$

$$\frac{\partial u}{\partial x} = -4ay\,e^{-2axy}\cos\left(ax^2-ay^2\right) - 4ax\,e^{-2axy}\sin\left(ax^2-ay^2\right)$$
$$\frac{\partial v}{\partial y} = -4ax\,e^{-2axy}\sin\left(ax^2-ay^2\right) - 4ay\,e^{-2axy}\cos\left(ax^2-ay^2\right) = \frac{\partial u}{\partial x}$$

Similarly, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, so that *f*(*z*) satisfies the Cauchy-Riemann relations and is therefore analytic.

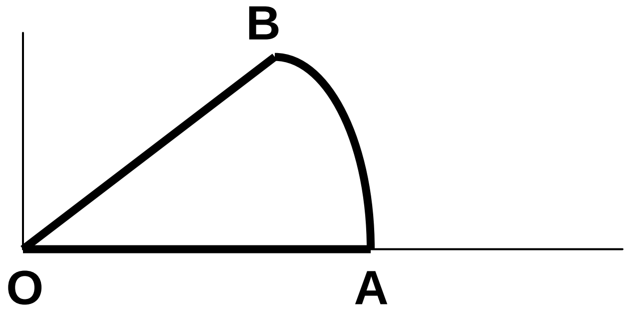
Now show that $f(z) \to 0$ as $|z| \to \infty$ for $0 < \arg z \le \pi/4$.

Let $z = r\,e^{i\theta} \Rightarrow z^2 = r^2 e^{i2\theta} = r^2 e^{i\left(\frac{\pi}{2}-\phi\right)} = r^2 e^{i\left(\frac{\pi}{2}\right)} e^{-i\phi}$ where $0 \le \phi < \frac{\pi}{2}$.

But $e^{i\left(\frac{\pi}{2}\right)} = i$ so that $f(z) = e^{iaz^2} = e^{-ar^2 e^{-i\phi}}$ which $\to 0$ *as* $r = |z| \to \infty$.

Note: this is true as long as $e^{-i\phi}$ has a real part: $\phi < \frac{\pi}{2}$ i.e. $\theta = \arg z > 0$.

The appropriate contour C is pie-shaped with an angle θ = π/4:



The integral consists of three pieces:

$$\oint_C \exp\left(iaz^2\right)dz = \int_O^A \exp\left(iaz^2\right)dz + \int_A^B \exp\left(iaz^2\right)dz + \int_B^O \exp\left(iaz^2\right)dz$$

The first integral is $\int_0^A \left(\cos ax^2 + i\sin ax^2\right)dx \underset{|z|\to\infty}{\rightarrow} \int_0^\infty \left(\cos ax^2 + i\sin ax^2\right)dx$

The second integral vanishes as $|z| \to \infty$ because $f(z) \to 0$ in this limit.

For the third integral, let $z=\rho e^{i\theta}$ and $dz = e^{i\theta}d\rho$ (at constant θ).

Along the line **O** – **B**, $\theta = \pi/4$ and $\exp\left(iaz^2\right) = \exp\left(ia\rho^2 e^{i\pi/2}\right) = \exp\left(-a\rho^2\right)$.

Then, the integral is

$$\int_B^O \exp\left(iaz^2\right)dz = e^{i\pi/4}\cdot\int_B^O \exp\left(-a\rho^2\right)d\rho = \left(\cos\pi/4 + i\sin\pi/4\right)\cdot\int_B^O \exp\left(-a\rho^2\right)d\rho\,.$$

In the limit $|z| \to \infty$, we have therefore

$$\oint_C \exp\left(iaz^2\right)dz = 0 \to \int_0^\infty \cos ax^2\,dx + i\int_0^\infty \sin ax^2\,dx + \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\cdot\int_\infty^0 \exp\left(-a\rho^2\right)d\rho = 0$$

Taking the real part and noting that $\int_0^\infty \exp\left(-a\rho^2\right)d\rho = \frac{1}{2}\frac{1}{\sqrt{a}}\Gamma\left(\frac{1}{2}\right) = \sqrt{\frac{\pi}{4a}}$,

we have $\int_0^\infty \cos ax^2 dx = \frac{1}{\sqrt{2}}\cdot\int_0^\infty \exp\left(-a\rho^2\right)d\rho = \frac{1}{\sqrt{2}}\cdot\sqrt{\frac{\pi}{4a}} = \sqrt{\frac{\pi}{8a}}$.