Problem Set #5

Solutions

1. RHB Problem 24.1
Note: Show, as suggested in the "Hints and Answers," that $f(z) = z \exp z$.

$$\operatorname{Im} f(z) = v(x,y) = (y\cos y + x\sin y)\exp x$$

Cauchy-Riemann: $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \qquad \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

Take the partial derivatives of v:

$$\frac{\partial v}{\partial x} = (\sin y + y\cos y + x\sin y)\exp x$$
$$\frac{\partial v}{\partial y} = (\cos y - y\sin y + x\cos y)\exp x$$

From the second Cauchy-Riemann relation $u(x,y) = -\int^y \dfrac{\partial v}{\partial x} dy$. Thus

$$u(x,y) = -\left(\int^y \sin y\, dy + \int^y y\cos y\, dy + x\int^y \sin y\, dy\right)\exp x$$
$$= -(-\cos y + \cos y + y\sin y - x\cos y)\exp x$$
$$u(x,y) = (-y\sin y + x\cos y)\exp x$$

Check:

$$\frac{\partial u}{\partial x} = (\cos y)\exp x + (-y\sin y + x\cos y)\exp x$$
$$= (\cos y - y\sin y + x\cos y)\exp x = \frac{\partial v}{\partial y}$$

Now,

$$\begin{aligned} f(z) &= u(x,y) + i\,v(x,y) \\ &= [(-y\sin y + x\cos y) + i(y\cos y + x\sin y)]\exp x \\ &= [y(-\sin y + i\cos y) + x(\cos y + i\sin y)]\exp x \\ &= [i\,y(\cos y + i\sin y) + x(\cos y + i\sin y)]\exp x \\ &= (x + i\,y)\exp i\,y \cdot \exp x = (x + i\,y)\exp(x + i\,y) \\ &= z\exp z \end{aligned}$$

2. RHB Problem 24.5.

(a) $f(z) = (z-2)^{-1} = \dfrac{1}{(z-2)}$.

This function is **analytic** at $z = 0$ and at $z = \infty$.

(b) $f(z) = \dfrac{(1+z^3)}{z^2}$

This function has a **double pole** at $z = 0$ and a **single pole** at $z = \infty$. To see that latter, let $\xi = 1/z$ and note that $f(1/\xi) = \xi^2 + 1/\xi$ has a single pole at $\xi = 0$, so that $f(z)$ has a single pole at ∞.

(c) $f(z) = \sinh(1/z)$

At z = 0,

$$\sinh(1/z) = e^{1/z} - e^{-1/z} = e^{1/z} - \frac{1}{e^{1/z}} \to e^{1/z} = \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{-n}}{n!}.$$

This series does not terminate at a finite negative power of z, so $z = 0$ is an **essential singularity**.

At $z = \infty$, consider $f(1/z) = f(\xi) = e^{\xi} - e^{-\xi}$ which is analytic at $\xi = 0$. Therefore, $f(z)$ is **analytic** at $z = \infty$.

(d) $f(z) = \dfrac{e^z}{z^3}$

At $z = 0$, $f(z)$ has a **triple pole**.

At $z = \infty$, consider $f(1/z) = f(\xi) = \xi^3 e^{1/\xi} = \xi^3 \cdot \sum_{n=0}^{\infty} \dfrac{\xi^{-n}}{n!} = \sum_{n=0}^{\infty} \dfrac{\xi^{-n+3}}{n!}$. The series does not terminate at a negative power of ξ, so $\xi = 0$ is an essential singularity and therefore $z = \infty$ is also an **essential singularity**.

(e) $f(z) = \frac{z^{1/2}}{(1+z^2)^{1/2}}$

As $z \to 0$, $f(z) \to z^{1/2}$. We have seen in class that $z^{1/2}$ has a **branch point** at $z = 0$.

As $\xi = 1/z \to 0$, $f(\xi) = \frac{\xi^{-1/2}}{(1+\xi^{-2})^{1/2}} \to \frac{\xi^{-1/2}}{\xi^{-1}} = \xi^{1/2}$. Again, this function has a branch point at $\xi = 0$ and therefore $z = \infty$ is also a **branch point**.

