# Problem Set # 3

## Solutions

RHB Problem 8.7

(a) $A$ is Hermetian and $U$ is unitary:

$$A^\dagger = A \qquad U^\dagger = U^{-1}$$

$$[U^{-1}AU]^\dagger = U^\dagger A^\dagger (U^{-1})^\dagger = U^{-1}A(U^\dagger)^{-1} = U^{-1}A(U^{-1})^{-1}$$
$$= \underline{U^{-1}AU}$$

(b) $A^\dagger = -A$

$$(iA)^\dagger = (-iA^*)^T = -i\,(A^*)^T = -i(-A) = \underline{iA}$$

(c) $A, B$ Hermetian

$$(AB)^\dagger = B^\dagger A^\dagger = BA = AB \quad \text{if + only if they commute}$$

i.e $(AB)^\dagger = AB$ if and only if $A, B$ commute

(d) If $A$ orthogonal, $A^TA = I$

$$A^TA = [(I-S)(I+S)^{-1}]^T[(I-S)(I+S)^{-1}]$$

$$= [(I+S)^{-1}]^T(I-S)^T(I-S)(I+S)^{-1}$$

$$(I-S)^T = I+S \qquad [(I+S)^{-1}]^T = (I-S)^{-1}$$

 

8.7-2

$$A^T A = (I-S)^{-1}(I+S)(I-S)(I+S)^{-1}$$

$$= (I-S)^{-1}(I-S+S-S^2)(I+S)^{-1}$$

$$= \underbrace{(I-S)^{-1}(I-S)}_{I}\underbrace{(I+S)(I+S)^{-1}}_{I} = I$$

$$A^T A = I$$

Now, note $A(I+S) = A + AS = (I-S)(I+S)^{-1}(I+S)$

$$= I - S$$

$$A + AS = I - S \Rightarrow AS + S = I - A$$

multiply by $(A+I)^{-1}$:

$$(A+I)^{-1}(AS+S) = (A+I^{-1})(A+I)S = (A+I)^{-1}(I-A)$$

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$S = (A+I)^{-1}(A+I) = \begin{pmatrix} 1+\cos\theta & \sin\theta \\ -\sin\theta & 1+\cos\theta \end{pmatrix}^{-1}\begin{pmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}$$

 

6.7-3

We need $\begin{pmatrix} 1+\cos\theta & \sin\theta \\ -\sin\theta & 1+\cos\theta \end{pmatrix}^{-1}$

$$\det(\;) = (1+\cos\theta)^2 + \sin^2\theta$$

$$= 1 + 2\cos\theta + \cos^2\theta + \sin^2\theta = 2(1+\cos\theta)$$

$$S = \frac{1}{2(1+\cos\theta)} \begin{pmatrix} 1+\cos\theta & -\sin\theta \\ \sin\theta & 1+\cos\theta \end{pmatrix} \begin{pmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix}$$

$$= \frac{1}{2(1+\cos\theta)} \begin{pmatrix} 1-\cos^2\theta-\sin^2\theta & \cancel{-\sin\theta-\cos\theta\sin\theta-\sin\theta+\cos\theta\sin\theta} \\ \sin\theta-\sin\theta\cos\theta+\sin\theta+\sin\theta\cos\theta & -\sin^2\theta+1-\cos^2\theta \end{pmatrix}$$

$$= \frac{1}{2(1+\cos\theta)} \begin{pmatrix} 0 & -2\sin\theta \\ 2\sin\theta & 0 \end{pmatrix}$$

$$\frac{\sin\theta}{1+\cos\theta} = \tan\theta/2$$

$$S = \begin{pmatrix} 0 & -\tan\theta/2 \\ \tan\theta/2 & 0 \end{pmatrix}$$

8.7-4

(d) $K^{\dagger} = -K \qquad V = (I+K)(I-K)^{-1}$ unitary?

$$V^{\dagger}V = \left[(I+K)(I-K)^{-1}\right]^{\dagger}\left[(I+K)(I-K)^{-1}\right]$$

$$= \left[(I-K)^{-1}\right]^{\dagger}(I+K)^{\dagger}(I+K)(I-K)^{-1}$$

$$(I+K)^{\dagger} = I-K \qquad \left[(I-K)^{-1}\right]^{\dagger} = (I+K)^{-1}$$

$$V^{\dagger}V = (I+K)^{-1}(I-K)(I+K)(I-K)^{-1}$$

$$= (I+K)^{-1}(I+K-K-K^2)(I-K)^{-1}$$

$$= \underbrace{(I+K)^{-1}(I+K)}_{I}\underbrace{(I-K)(I-K)^{-1}}_{I} = I$$

$$V^{\dagger}V = I$$

 

2. RHB Problem 8.16.

The given matrix is $\mathbf{A} = \begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{pmatrix}$ and the secular equation for the eigenvalues is

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 3 & -1 \\ 3 & 4-\lambda & -2 \\ -1 & -2 & 2-\lambda \end{vmatrix} = 0$$

Evaluating the determinant by minors,

$$\begin{aligned} |\mathbf{A} - \lambda \mathbf{I}| &= (1-\lambda)\begin{vmatrix} 4-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & -2 \\ -1 & 2-\lambda \end{vmatrix} - \begin{vmatrix} 3 & 4-\lambda \\ -1 & -2 \end{vmatrix} \\ &= (1-\lambda)[(4-\lambda)(2-\lambda)-4] - 3[3(2-\lambda) - 2] - [-6 + (4-\lambda)] \\ &= (1-\lambda)\left[\lambda^2 - 6\lambda - 6\right] = 0 \end{aligned}$$

The first factor gives immediately $\lambda_1 = 1$ while the quadratic gives

$$\lambda_{\substack{2\\3}} = \frac{6 \pm \sqrt{36+24}}{2} = 3 \pm \sqrt{15}$$

Check the eigenvalues: $Tr\, \mathbf{A} = 1 + 4 + 2 = 7 = 1 + \left(3 + \sqrt{15}\right) + \left(3 - \sqrt{15}\right) = \lambda_1 + \lambda_2 + \lambda_3$

The eigenvector for $\lambda_1 = 1$ is given by

$$\begin{pmatrix} 1-1 & 3 & -1 \\ 3 & 4-1 & -2 \\ -1 & -2 & 2-1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -1 \\ 3 & 3 & -2 \\ -1 & -2 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0\,.$$

Multiplying out the first two rows of the matrix, $\begin{aligned} 3c_2 - c_3 &= 0 \\ 3c_1 + 3c_2 - 2c_3 &= 0 \end{aligned}$.

Expressing $c_2$ and $c_3$ in terms of $c_1$, we have $c_3 = 3c_1$ and $c_2 = \frac{1}{3}c_3 = c_1$. Thus the eigenvector is

$\mathbf{x}^1 = c_1\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \frac{1}{\sqrt{11}}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ when normalized.

(Normalization is not required.)

 

The eigenvector for $\lambda_2 = 3 + \sqrt{15}$ is given by

$$\begin{pmatrix} 1-3-\sqrt{15} & 3 & -1 \\ 3 & 4-3-\sqrt{15} & -2 \\ -1 & -2 & 2-3-\sqrt{15} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2-\sqrt{15} & 3 & -1 \\ 3 & 1-\sqrt{15} & -2 \\ -1 & -2 & -1-\sqrt{15} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

Multiplying out the first two rows of the matrix,

$$\begin{aligned} -\left(2+\sqrt{15}\right)c_1 + 3c_2 - c_3 &= 0 \\ 3c_1 + \left(1-\sqrt{15}\right)c_2 - 2c_3 &= 0 \end{aligned} \qquad \Rightarrow \qquad c_2 = \frac{7+2\sqrt{15}}{5+\sqrt{15}}c_1,\;\; c_3 = -\frac{4+\sqrt{15}}{5+\sqrt{15}}c_1$$

The eigenvector is $\mathbf{x}^2 = \dfrac{1}{\sqrt{180+46\sqrt{15}}}\begin{pmatrix} 5+\sqrt{15} \\ 7+2\sqrt{15} \\ -4-\sqrt{15} \end{pmatrix}$ if normalized.

The eigenvector for $\lambda_3 = 3 - \sqrt{15}$ is given by

$$\begin{pmatrix} 1-3+\sqrt{15} & 3 & -1 \\ 3 & 4-3+\sqrt{15} & -2 \\ -1 & -2 & 2-3+\sqrt{15} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -2+\sqrt{15} & 3 & -1 \\ 3 & 1+\sqrt{15} & -2 \\ -1 & -2 & -1+\sqrt{15} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0.$$

Multiplying out the first two rows of the matrix,

$$\begin{aligned} -\left(2-\sqrt{15}\right)c_1 + 3c_2 - c_3 &= 0 \\ 3c_1 + \left(1+\sqrt{15}\right)c_2 - 2c_3 &= 0 \end{aligned} \qquad \Rightarrow \qquad c_2 = \frac{7-2\sqrt{15}}{5-\sqrt{15}}c_1,\;\; c_3 = \frac{-4+\sqrt{15}}{5-\sqrt{15}}c_1$$

The eigenvector is $\mathbf{x}^3 = \dfrac{1}{\sqrt{180-46\sqrt{15}}}\begin{pmatrix} 5-\sqrt{15} \\ 7-2\sqrt{15} \\ -4+\sqrt{15} \end{pmatrix}$ if normalized.

 

To verify orthogonality, look at the inner products (can neglect normalization factors):

$$\langle \mathbf{x}^2 | \mathbf{x}^1 \rangle = \begin{pmatrix} 5+\sqrt{15} & 7+2\sqrt{15} & -4-\sqrt{15} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$= 5+\sqrt{15}+7+2\sqrt{15}-12-3\sqrt{15}=0$$

$$\langle \mathbf{x}^3 | \mathbf{x}^1 \rangle = \begin{pmatrix} 5-\sqrt{15} & 7-2\sqrt{15} & -4+\sqrt{15} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$= 5-\sqrt{15}+7-2\sqrt{15}-12+3\sqrt{15}=0$$