Problem Set #2

Solutions

1. RHB Problem 8.6.

(a) $G_{ij}^{-1} = H_{ij}$ where $G_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ and $\mathbf{f}_i = \sum_j H_{ij}\mathbf{e}_j$

$$\mathbf{f}_i \cdot \mathbf{e}_k = \sum_j H_{ij}\mathbf{e}_j \cdot \mathbf{e}_k = \sum_j H_{ij}G_{jk} = (\mathbf{HG})_{ik} = \left(\mathbf{G}^{-1}\mathbf{G}\right)_{ik} = \delta_{ik}$$

Thus the $\mathbf{f}_i$ are the reciprocal vectors: $\mathbf{f}_1 \cdot \mathbf{e}_1 = 1 \quad \mathbf{f}_1 \cdot \mathbf{e}_2 = 0$, etc.

$$\mathbf{f}_i \cdot \mathbf{f}_j = \sum_k H_{ik}\mathbf{e}_k \cdot \sum_\ell H_{j\ell}\mathbf{e}_\ell = \sum_{k,\ell} H_{ik}H_{j\ell}\,\mathbf{e}_k \cdot \mathbf{e}_\ell = \sum_k H_{ik}\sum_\ell H_{j\ell}G_{\ell k} = \sum_k H_{ik}\left(\mathbf{G}^{-1}\mathbf{G}\right)_{jk} = H_{ij}$$

(b) $\mathbf{u} = \sum_i u_i\mathbf{e}_i \qquad \mathbf{v} = \sum_i v_i\mathbf{f}_i \qquad$ and use $\langle \mathbf{a}|\mathbf{b}\rangle = \sum_i\sum_j a_i^* G_{ij}\, b_j$

$$|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}} = \left(\sum_{i,j} u_i G_{ij} u_j\right)^{1/2} \qquad |\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \left(\sum_{i,j} v_i H_{ij} v_j\right)^{1/2}$$

$$\mathbf{u}\cdot\mathbf{v} = \sum_i u_i\mathbf{e}_i \cdot \sum_j v_j\mathbf{f}_j = \sum_{i,j} u_i v_j \mathbf{e}_i \cdot \mathbf{f}_j = \sum_{i,j} u_i v_j \delta_{ij} = \sum_i u_i v_i$$

(c) $$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{matrix} a^2\cos\dfrac{\pi}{3} = \dfrac{a^2}{2} & i \neq j \\ a^2 & i = j \end{matrix} \qquad \text{and} \quad G = a^2\begin{pmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{pmatrix}$$

$$H_{ik} = \frac{C_{ki}}{|\mathbf{G}|} \quad \text{where} \quad |\mathbf{G}| = a^6\left[1\cdot\left(1-\frac{1}{4}\right) - \frac{1}{2}\cdot\left(\frac{1}{2}-\frac{1}{4}\right) + \frac{1}{2}\cdot\left(\frac{1}{4}-\frac{1}{2}\right)\right] = \frac{a^6}{2}$$

The cofactors are:

$$C_{11} = \left(1-\frac{1}{4}\right) = \frac{3a^4}{4} \qquad C_{12} = -\left(\frac{1}{2}-\frac{1}{4}\right) = -\frac{a^4}{4} \qquad C_{13} = \left(\frac{1}{4}-\frac{1}{2}\right) = -\frac{a^4}{4}$$

$$C_{21} = -\left(\frac{1}{2}-\frac{1}{4}\right) = -\frac{a^4}{4} \qquad C_{22} = \left(1-\frac{1}{4}\right) = \frac{3a^4}{4} \qquad C_{23} = -\left(\frac{1}{2}-\frac{1}{4}\right) = -\frac{a^4}{4}$$

$$C_{31} = \left(\frac{1}{4}-\frac{1}{2}\right) = -\frac{a^4}{4} \qquad C_{32} = -\left(\frac{1}{2}-\frac{1}{4}\right) = -\frac{a^4}{4} \qquad C_{33} = \left(1-\frac{1}{4}\right) = \frac{3a^4}{4}$$

$$H_{11} = H_{22} = H_{33} = \frac{3a^4\!\big/4}{a^6\!\big/2} = \frac{1}{a^2}\frac{3}{2}$$

Then,

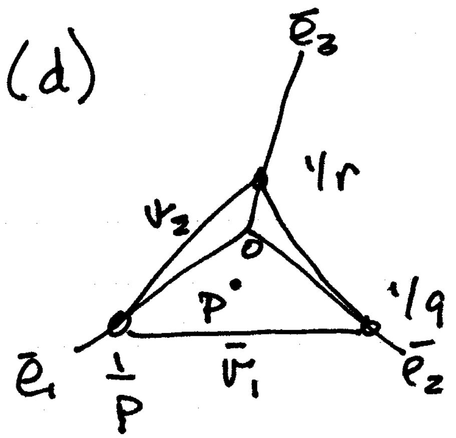
$$H_{12} = H_{13} = H_{21} = H_{23} = H_{31} = H_{32} = \frac{-a^4\!\big/4}{a^6\!\big/2} = -\frac{1}{a^2}\frac{1}{2}$$

$$\mathbf{H} = \frac{1}{a^2}\begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

Check:

$$\mathbf{HG} = \mathbf{G}^{-1}\mathbf{G} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}$$

(d)



Let $\bar{n}$ = vector from O to P in plane and perpendicular to plane

$$\bar{n} = n_1\bar{f}_1 + n_2\bar{f}_2 + n_3\bar{f}_3 \quad (\text{use } \bar{f}_i \text{ basis})$$

Let $\bar{v}_1$ and $\bar{v}_2$ be vectors $1/p \to 1/q$ and $1/q - 1/r$, respectively. These vectors define the plane.

If $\bar{n} \perp$ to plane, we must have

$$\bar{n}\cdot\bar{v}_1 = \bar{n}\cdot\bar{v}_2 = 0$$

$$\bar{v}_1 = -\frac{\bar{e}_1}{p} + \frac{\bar{e}_2}{q} \qquad \bar{v}_2 = -\frac{\bar{e}_1}{p} + \frac{\bar{e}_3}{r}$$

$$\bar{n}\cdot\bar{v}_1 = (n_1\bar{f}_1 + n_2\bar{f}_2 + n_3\bar{f}_3)\cdot\left(-\frac{\bar{e}_1}{p} + \frac{\bar{e}_2}{q}\right)$$

$$= -\frac{n_1}{p} + \frac{n_2}{q} \quad \text{since } e_i\cdot f_j = \delta_{ij} \text{ (reciprocals)}$$

Similarly, $\bar{n}\cdot\bar{v}_2 = -\frac{n_1}{p} + \frac{n_3}{r}$

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$\bar{n}\cdot\bar{v}_1 = 0 \Rightarrow n_2 = \frac{q}{p} n_1$

$\bar{n}\cdot\bar{v}_2 = 0 \Rightarrow n_3 = \frac{r}{p} n_1$

Thus, $\bar{n} = n_0 (p\bar{f}_1 + q\bar{f}_2 + r\bar{f}_3)$

$\bar{n}$ is $\perp$ to plane, but we must adjust length (set n_0) so that P lies in plane.

Consider the vector $\bar{v}_3$ from $\frac{1}{p}\bar{e}_1$ to P:

$\bar{v}_3 = n - \frac{\bar{e}_1}{p}$ and we must have $\bar{n}\cdot\bar{v}_3 = 0$

$$\bar{n}\cdot\bar{v}_3 = \bar{n}\cdot\bar{n} - \frac{1}{p}\bar{n}\cdot\bar{e}_1$$

$$\bar{n}\cdot\bar{n} = |n|^2 = \sum_{i,j} n_i H_{ij} n_j \qquad \text{(from part b)}$$

$$= \frac{n_0^2}{a^2}\left[\frac{3}{2}(p^2+q^2+r^2) - \frac{1}{2}(2pq + 2pr + 2qr)\right]$$

$$= \frac{n_0^2}{2a^2}\left[3(p^2+q^2+r^2) - 2pq - 2pr - 2qr\right]$$

$$\equiv \frac{n_0^2}{2a^2} c(p,q,r)$$

where $c(p,q,r) = 3(p^2+q^2+r^2) - 2(pq+pr+qr)$

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Then,

$$\bar{n}\cdot v_3 = \frac{n_0^2}{2a^2}c(p,q,r) - n_0 p\bar{f}_1 \cdot \frac{1}{p}\bar{e}_1$$

$$= \frac{n_0^2}{2a^2}c(p,q,r) - n_0 = 0$$

so $n_0 = \dfrac{2a^2}{c(p,q,r)}$ and

$$\bar{n} = \frac{2a^2}{c(p,q,r)}\left(p\bar{f}_1 + q\bar{f}_2 + r\bar{f}_3\right)$$

The distance O-P is $|n| = (\bar{n}\cdot\bar{n})^{1/2}$

$$= \frac{n_0}{\sqrt{2}a}[c(p,q,r)]^{1/2} = \frac{[\;]^{1/2}}{\sqrt{2}a}\cdot\frac{2a^2}{[\;]} = \frac{\sqrt{2}a}{[c(p,q,r)]^{1/2}}$$

To find the angle with $\bar{e}_1$, consider the unit vectors

$$\hat{e}_1 = \frac{\bar{e}_1}{a} \qquad \hat{n} = \frac{1}{|n|}\bar{n} = \frac{[c(p,q,r)]^{1/2}}{\sqrt{2}a}\cdot\frac{2a^2}{c(p,q,r)}\left(p\bar{f}_1 + q\bar{f}_2 + r\bar{f}_3\right)$$

$$\hat{n} = \frac{\sqrt{2}a}{[c(p,q,r)]^{1/2}}\left(p\bar{f}_1 + q\bar{f}_2 + r\bar{f}_3\right)$$

$$\cos\theta = \hat{e}\cdot\hat{n} = \frac{\sqrt{2}p}{[c(p,q,r)]^{1/2}}$$

2. (i) Find the inverse of the following matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ -2 & -3 & 1 \end{pmatrix}$$

First evaluate the determinant of the matrix **A**:

$$\det \mathbf{A} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ -2 & -3 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix} - 2\cdot\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} + 3\cdot\begin{vmatrix} 2 & 0 \\ -2 & -3 \end{vmatrix}$$

$$= 1\cdot(-3) - 2\cdot(0) + 3\cdot(-6) = -3 - 18 = -21$$

Then calculate the cofactor matrix:

$$C = \begin{pmatrix} \begin{vmatrix} 0 & -1 \\ -3 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ -2 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ -2 & -3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -2 & -3 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -3 & 0 & -6 \\ -11 & 7 & -1 \\ -2 & 7 & -4 \end{pmatrix}$$

Finally, the inverse matrix is

$$\mathbf{A}^{-1} = \frac{\mathbf{C}^{\mathrm{T}}}{|\mathbf{A}|} = -\frac{1}{21}\begin{pmatrix} -3 & -11 & -2 \\ 0 & 7 & 7 \\ -6 & -1 & -4 \end{pmatrix}$$

(ii) Verify by direct matrix multiplication that your inverse matrix is correct.

$$\mathbf{A}^{-1}\mathbf{A} = \left(\frac{-1}{21}\right)\begin{pmatrix} -3 & 11 & -2 \\ 0 & 7 & 7 \\ -6 & -1 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ -2 & -3 & 1 \end{pmatrix} = \left(\frac{-1}{21}\right)\begin{pmatrix} -3-22+4 & -6+0+6 & -9+11-2 \\ 0+14-14 & 0+0-21 & 0-7+7 \\ -6-2+8 & -12+0+12 & -18+1-4 \end{pmatrix}$$

$$= \left(\frac{-1}{21}\right)\begin{pmatrix} -21 & 0 & 0 \\ 0 & -21 & 0 \\ 0 & 0 & -21 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

