Problem Set #1

Solutions

1. (i) Normalize the following 4-dimensional vectors and show that they form an orthogonal set:

$\mathbf{a} = (2 \quad 2 \quad 2 \quad 2)^T$ $\mathbf{b} = (1 \quad 1 \quad -1 \quad -1)^T$ $\mathbf{c} = (-3 \quad 3 \quad 3 \quad -3)^T$ $\mathbf{d} = (2 \quad -2 \quad 2 \quad -2)^T$

Determine the norms of the vectors and divide:

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} = \sqrt{4 + 4 + 4 + 4} = 4$$

$$\hat{\mathbf{a}} = \frac{1}{4}(2 \quad 2 \quad 2 \quad 2)^T = \left(\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2}\right)^T$$

$$\|\mathbf{b}\| = 2 \qquad \hat{\mathbf{b}} = \left(\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad -\frac{1}{2}\right)^T$$

Similarly, $\|\mathbf{c}\| = 6 \qquad \hat{\mathbf{c}} = \left(-\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2}\right)^T$

$$\|\mathbf{d}\| = 4 \qquad \hat{\mathbf{d}} = \left(\frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2}\right)^T$$

To show orthogonality, show that the inner products vanish. For example,

$$\langle \hat{\mathbf{a}} | \hat{\mathbf{b}} \rangle = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = 0$$

$$\langle \hat{\mathbf{a}} | \hat{\mathbf{c}} \rangle = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} = 0$$

(ii) Express the vector $\mathbf{x}$ = (4, 2, –1, 1) in the orthonormal basis consisting of the normalized vectors you found in part (i).

To get the coefficients, project $\mathbf{x}$ onto the basis vectors:

$$\langle \mathbf{a} | \mathbf{x} \rangle = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -1 \\ 1 \end{pmatrix} = 2 + 1 - \frac{1}{2} + \frac{1}{2} = 3$$

Similarly, $\langle \mathbf{b} | \mathbf{x} \rangle = 3$ $\langle \mathbf{c} | \mathbf{x} \rangle = -2$ $\langle \mathbf{d} | \mathbf{x} \rangle = 0$

Therefore, $\mathbf{x} = 3\hat{\mathbf{a}} + 3\hat{\mathbf{b}} - 2\hat{\mathbf{c}}$.

2. Carry out the following matrix multiplication:

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$$

First, multiply the two matrices on the left,

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & -4 \\ 0 & 4 & 3 \end{pmatrix}\begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 9 & -8 & 0 \\ 0 & 12 & 6 & 0 \end{pmatrix}$$

Then,

$$\begin{pmatrix} 20 & 0 & 0 & 0 \\ 0 & 9 & -8 & 0 \\ 0 & 12 & 6 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 20 & 20 & 20 & 20 \\ 9+8 & 9-8 & -9-8 & -9+8 \\ 12-6 & 12+6 & -12+6 & -12-6 \end{pmatrix}$$

$$= \begin{pmatrix} 20 & 20 & 20 & 20 \\ 17 & 1 & -17 & -1 \\ 6 & 18 & -6 & -18 \end{pmatrix}$$

