THE GEOMETRY OF SPECIAL RELATIVITY

Tevian Dray
Department of Mathematics, Oregon State University
tevian@math.orst.edu

DRAFT, October 4, 2002

Lorentz transformations are just hyperbolic rotations.

Chapter 4

Hyperbola Geometry

In which a 2-dimensional non-Euclidean geometry is constructed, which will turn out to be identical with special relativity.

4.1 Trigonometry

The hyperbolic trig functions are usually defined using the formulas

$$\cosh\beta = \frac{e^{\beta}+e^{-\beta}}{2} \tag{4.1}$$

$$\sinh\beta = \frac{e^{\beta}-e^{-\beta}}{2} \tag{4.2}$$

and then

$$\tanh\beta = \frac{\sinh\beta}{\cosh\beta} \tag{4.3}$$

and so on. We will discuss an alternative definition below. The graphs of these functions are shown in Figure 4.1.

It is straightforward to verify from these definitions that

$$\cosh^2\beta - \sinh^2\beta = 1 \tag{4.4}$$

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta \tag{4.5}$$

$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \tag{4.6}$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha+\tanh\beta}{1+\tanh\alpha\tanh\beta} \tag{4.7}$$

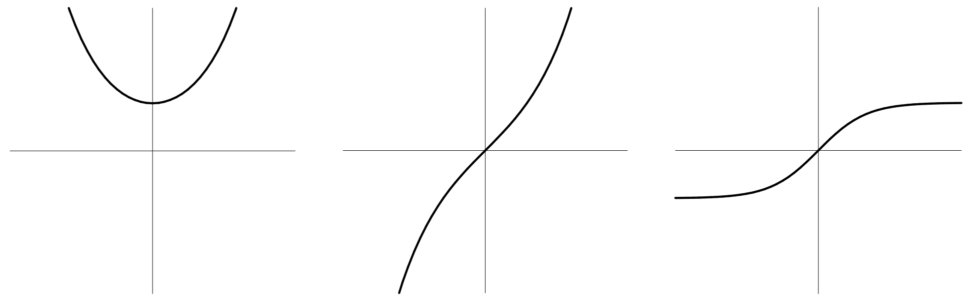

Figure 4.1: The graphs of $\cosh\beta$, $\sinh\beta$, and $\tanh\beta$, respectively.

$$\frac{d}{d\beta}\sinh\beta = \cosh\beta \tag{4.8}$$

$$\frac{d}{d\beta}\cosh\beta = \sinh\beta \tag{4.9}$$

These hyperbolic trig identities look very much like their ordinary trig counterparts (except for signs). This similarity derives from the fact that

$$\cosh\beta \equiv \cos(i\beta) \tag{4.10}$$

$$\sinh\beta \equiv -i\sin(i\beta) \tag{4.11}$$

4.2 Distance

We saw in the last chapter that Euclidean distance is based on the *unit circle*, the set of points which are unit distance from the origin. Hyperbola geometry is obtained simply by using a different distance function! Measure the "squared distance" of a point $B = (x, y)$ from the origin using the definition

$$\delta^2 = x^2 - y^2 \tag{4.12}$$

Then the unit "circle" becomes the unit hyperbola

$$x^2 - y^2 = 1 \tag{4.13}$$

and we further restrict ourselves to the branch with $x > 0$. If B is a point on this hyperbola, then we can *define* the hyperbolic angle β between the line from the origin to B and the (positive) x-axis to be the *Lorentzian* length [1]

[1] No, we haven't defined this. In Euclidean geometry, the the length of a curve is obtained by integrating ds along the curve, where $ds^2 = dx^2 + dy^2$. In a similar way, the Lorentzian length is obtained by integrating $d\sigma$, where $d\sigma^2 = dx^2 - dy^2$.

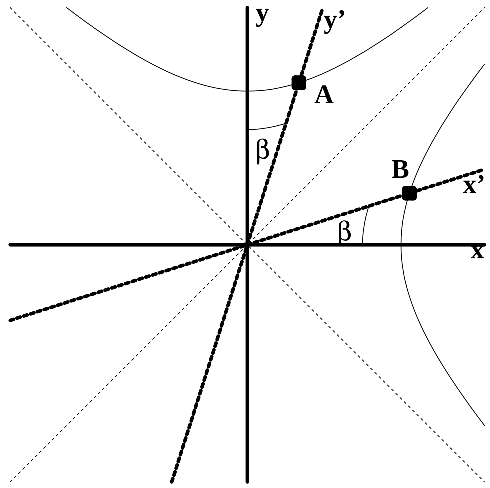

Figure 4.2: The unit hyperbola. The point A has coordinates $(\sinh\beta, \cosh\beta)$, and $B = (\cosh\beta, \sinh\beta)$.

of the arc of the unit hyperbola between B and the point $(1, 0)$. We could then *define* the hyperbolic trig functions to be the coordinates (x, y) of B, that is

$$\cosh\beta = x \tag{4.14}$$
$$\sinh\beta = y \tag{4.15}$$

and a little work shows that this definition is exactly the same as the one above. [2] This construction is shown in Figure 4.2, which also shows another "unit" hyperbola, given by $x^2 - y^2 = -1$. By symmetry, the point A on this hyperbola has coordinates $(x, y) = (\sinh\beta, \cosh\beta)$. We will discuss the importance of this hyerbola later.

Many of the features of the graphs shown in Figure 4.1 follow immediately from this definition of the hyperbolic trig functions in terms of coordinates along the unit hyperbola. Since the minimum value of x on this hyperbola

[2]Use $x^2 - y^2 = 1$ to compute

$$d\beta^2 \equiv d\sigma^2 = dy^2 - dx^2 = \frac{dx^2}{x^2 - 1} = \frac{dy^2}{y^2 + 1}$$

then take the square root of either expression and integrate. (The integrals are hard.) Finally, solve for x or y in terms of β, yielding (4.1) or (4.2), respectively.

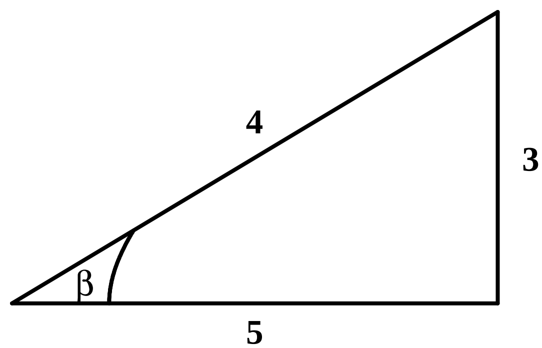

Figure 4.3: A hyperbolic triangle with $\tanh\beta = \frac{3}{5}$.

is 1, we must have $\cosh\beta \geq 1$. As β approaches $\pm\infty$, x approaches ∞ and y approaches $\pm\infty$, which agrees with the asymptotic behavior of the graphs of $\cosh\beta$ and $\sinh\beta$, respectively. Finally, since the hyperbola has asymptotes $y = \pm x$, we see that $|\tanh\beta| < 1$, and that $\tanh\beta$ must approach ± 1 as β approaches $\pm\infty$.

So how do we measure the distance between two points? The "squared distance" was defined in (4.12), and can be positive, negative, or zero! We adopt the following convention: *Take the square root of the absolute value of the "squared distance".* As we will see in the next chapter, it will also be important to remember whether the "squared distance" was positive or negative, but this corresponds directly to whether the distance is "mostly horizontal" or "mostly vertical".

4.3 Triangle Trig

We now recast ordinary triangle trig into hyperbola geometry.

Suppose you know $\tanh\beta = \frac{3}{5}$, and you wish to determine $\cosh\beta$. One can of course do this algebraically, using the identity

$$\cosh^2\beta = \frac{1}{1 - \tanh^2\beta} \tag{4.16}$$

But it is easier to draw *any* triangle containing an angle whose hyperbolic tangent is $\frac{3}{5}$. In this case, the obvious choice would be the triangle shown in Figure 4.3, with sides of 3 and 5.

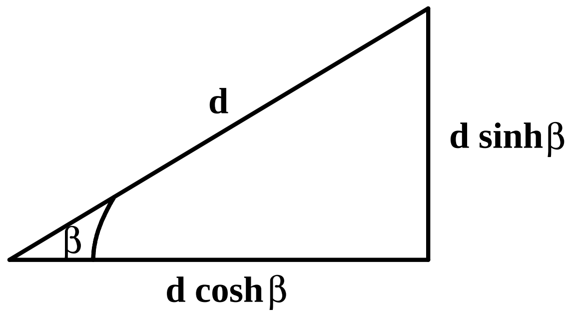

Figure 4.4: A hyperbolic triangle in which the hypotenuse and one angle are known.

What is $\cosh\beta$? Well, we first need to work out the length δ of the hypotenuse. The (hyperbolic) Pythagorean Theorem tells us that

$$5^2 - 3^2 = \delta^2 \tag{4.17}$$

so δ is clearly 4. Take a good look at this 3-4-5 triangle of hyperbola geometry, which is shown in Figure 4.3! But now that we know all the sides of the triangle, it is easy to see that $\cosh\beta = \frac{5}{4}$.

Trigonometry is not merely about ratios of sides, it is also about projections. Another common use of triangle trig is to determine the sides of a triangle given the hypotenuse d and one angle β. The answer, of course, is that the sides are $d\cosh\beta$ and $d\sinh\beta$, as shown in in Figure 4.4.

4.4 Rotations

By analogy with the Euclidean case, we *define* a hyperbolic rotation through the relations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{4.18}$$

This corresponds to “rotating” both the x and y axes into the first quadrant, as shown in Figure 4.2. While this may seem peculiar, it is easily verified that the “distance” is invariant, that is,

$$x^2 - y^2 \equiv x'^2 - y'^2 \tag{4.19}$$

which follows immediately from the hyperbolic trig identity (4.4).

4.5 Projections

We can ask the same question as we did for Euclidean geometry. Consider a rectangle of width 1 whose sides are parallel to the unprimed axes. How wide is it when measured in the primed coordinates? It turns out that the width of the box in the primed coordinate system is *less than* 1. This is length contraction, to which we will return in the next chapter, along with time dilation.

4.6 Addition Formulas

What is the slope of the line from the origin to the point A in Figure 4.2? The equation of this line, the y'-axis, is

$$x = y \tanh \beta \tag{4.20}$$

Consider now a line with equation

$$x' = y' \tanh \alpha \tag{4.21}$$

What is its (unprimed) slope? Again, slopes don't add, but (hyperbolic) angles do; the answer is that

$$x = y \tanh(\alpha + \beta) \tag{4.22}$$

which can be expressed in terms of the slopes $\tanh \alpha$ and $\tanh \beta$ using (4.7). As discussed in more detail in the next chapter, this is the Einstein addition formula!

Chapter 5

The Geometry of Special Relativity

In which it is shown that special relativity is just hyperbolic geometry.

5.1 Spacetime Diagrams

A brilliant aid in understanding special relativity is the *Surveyor's parable* introduced by Taylor and Wheeler [1, 2]. Suppose a town has daytime surveyors, who determine North and East with a compass, nighttime surveyors, who use the North Star. These notions of course differ, since magnetic north is not the direction to the North Pole. Suppose further that both groups measure north/south distances in miles and east/west distances in meters, with both being measured from the town center. How does one go about comparing the measurements of the two groups?

With our knowledge of Euclidean geometry, we see how to do this: Convert miles to meters (or vice versa). Furthermore, distances computed with the Pythagorean theorem do not depend on which group does the surveying. Finally, it is easily seen that "daytime coordinates" can be obtained from "nighttime coordinates" by a simple rotation. The moral of this parable is therefore:

1. *Use the same units.*
2. *The (squared) distance is invariant.*
3. *Different frames are related by rotations.*

Applying that lesson to relativity, the first thing to do is to measure both time and space in the same units. How does one measure distance in seconds? that's easy: simply multiply by c. Thus, since $c = 3 \times 10^8 \ \frac{\text{m}}{\text{s}}$, 1 second of distance is just 3×10^8 m. [1] Note that this has the effect of setting $c = 1$, since the number of seconds (of distance) traveled by light in 1 second (of time) is precisely 1.

Of course, it is also possible to measure time in meters: simply divide by c. Thus, 1 meter of time is the time it takes for light (in vacuum) to travel 1 meter. Again, this has the effect of setting $c = 1$.

5.2 Lorentz Transformations

The Lorentz transformation between a frame (x, t) at rest and a frame (x', t') moving to the right at speed v was derived in Chapter 2. The transformation from the moving frame to the frame at rest is given by

$$x = \gamma\,(x' + vt') \tag{5.1}$$

$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{5.2}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{5.3}$$

The key to converting this to hyperbola geometry is to measure space and time in the same units by replacing t by ct. The transformation from the moving frame, which we now denote (x', ct'), to the frame at rest, now denoted (x, ct), is given by

$$x = \gamma\,(x' + \frac{v}{c}\,ct') \tag{5.4}$$

$$ct = \gamma\left(ct' + \frac{v}{c}\,x'\right) \tag{5.5}$$

which makes the symmetry between these equations much more obvious.

We can simplify things still further. Introduce the *rapidity* β via [2]

$$\frac{v}{c} = \tanh\beta \tag{5.6}$$

[1] A similar unit of distance is the *lightyear*, namely the distance traveled by light in 1 year, which would here be called simply a *year* of distance.

[2] WARNING: Some authors use β for $\frac{v}{c}$, not the rapidity.

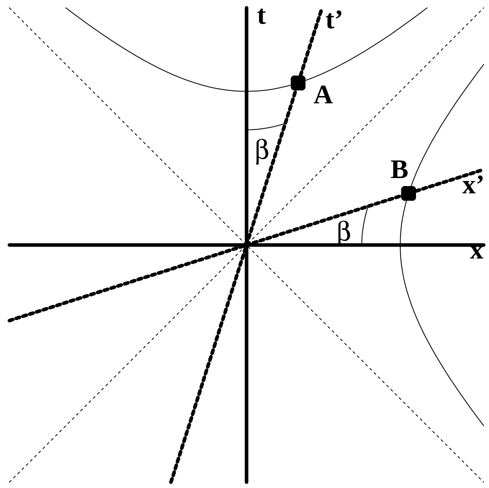

Figure 5.1: The Lorentz transformation between an observer at rest and an observer moving at speed $\frac{v}{c} = c \tanh\beta$ is shown as a hyperbolic rotation. The point A has coordinates $(\sinh\beta, \cosh\beta)$, and $B = (\cosh\beta, \sinh\beta)$. (Units have been chosen such that $c = 1$.)

Inserting this into the expression for γ we obtain

$$\gamma = \frac{1}{\sqrt{1 - \tanh^2\beta}} = \sqrt{\frac{\cosh^2\beta}{\cosh^2\beta - \sinh^2\beta}} = \cosh\beta \tag{5.7}$$

and

$$\frac{v}{c}\gamma = \tanh\beta\cosh\beta = \sinh\beta \tag{5.8}$$

Inserting these identities into the Lorentz transformations above brings them to the remarkably simple form

$$x = x'\cosh\beta + ct'\sinh\beta \tag{5.9}$$

$$ct = x'\sinh\beta + ct'\cosh\beta \tag{5.10}$$

which in matrix form are just

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{5.11}$$

But (5.11) is just (4.18), with $y = ct$!

Chapter 6

Applications

6.1 Addition of Velocities

What is the rapidity β? Consider an observer moving at speed v to the right. This observer's world line intersects the unit hyperbola

$$c^2t^2 - x^2 = 1 \qquad (ct > 0) \tag{6.1}$$

at the point $A = (\sinh\beta, \cosh\beta)$; this line has "slope" [1]

$$\frac{v}{c} = \tanh\beta \tag{6.2}$$

as required. Thus, β can be thought of as the *hyperbolic angle* between the ct-axis and the worldline of a moving object. As discussed in the preceding chapter, β turns out to be precisely the distance from the axis as measured along the hyperbola (in hyperbola geometry!). This was illustrated in Figure **??**.

Consider therefore an object moving at speed u relative to an observer moving at speed v. Their rapidities are given by

$$\frac{u}{c} = \tanh\alpha \tag{6.3}$$

$$\frac{v}{c} = \tanh\beta \tag{6.4}$$

[1]It is not obvious whether "slope" should be defined by $\frac{\Delta x}{c\,\Delta t}$ or by the reciprocal of this expression. This is further complicated by the fact that both (x, ct) and (ct, x) are commonly used to denote the coordinates of the point A!

To determine the resulting speed with respect to an observer at rest, simply add the *rapidities*! One way to think of this is that you are adding the arc lengths along the hyperbola. Another is that you are following a (hyperbolic) rotation through a (hyperbolic) angle β (to get to the moving observer's frame) with a rotation through an angle α. In any case, the resulting speed w is given by

$$\frac{w}{c} = \tanh(\alpha + \beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}} \tag{6.5}$$

which is — finally — precisely the Einstein addition formula!