THE GEOMETRY OF SPECIAL RELATIVITY

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Lorentz transformations are just hyperbolic rotations.

1 Hyperbola Geometry

In which a 2-dimensional non-Euclidean geometry is constructed, which will turn out to be identical with special relativity.

1.1 Trigonometry

The hyperbolic trig functions are usually defined using the formulas

$$\cosh\beta = \frac{e^\beta + e^{-\beta}}{2} \tag{1}$$

$$\sinh\beta = \frac{e^\beta - e^{-\beta}}{2} \tag{2}$$

and then

$$\tanh\beta = \frac{\sinh\beta}{\cosh\beta} \tag{3}$$

and so on. We will discuss an alternative definition below. It is straightforward to verify from these definitions that

$$\cosh^2\beta - \sinh^2\beta = 1 \tag{4}$$

$$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta \tag{5}$$

$$\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta \tag{6}$$

$$\tanh(\alpha+\beta) = \frac{\tanh\alpha + \tanh\beta}{1+\tanh\alpha\tanh\beta} \tag{7}$$

$$\frac{d}{d\beta}\sinh\beta = \cosh\beta \tag{8}$$

$$\frac{d}{d\beta}\cosh\beta = \sinh\beta \tag{9}$$

These hyperbolic trig identities look very much like their ordinary trig counterparts (except for signs). This similarity derives from the fact that

$$\cosh\beta \equiv \cos(i\beta) \tag{10}$$

$$\sinh\beta \equiv -i\sin(i\beta) \tag{11}$$

1.2 Distance

Euclidean distance is based on the *unit circle*, the set of points which are unit distance from the origin. Hyperbola geometry is obtained simply by using a different distance function! Measure the (squared) "distance" of a point $B = (x, y)$ from the origin using the definition

$$\delta^2 = x^2 - y^2 \tag{12}$$

Then the unit "circle" becomes the unit hyperbola

$$x^2 - y^2 = 1 \tag{13}$$

and we further restrict ourselves to the branch with $x > 0$. If B is a point on this hyperbola, then we can *define* the hyperbolic angle β between the line from the origin to B and the (positive) x-axis to be the *Lorentzian* length [1] of the arc of the unit hyperbola between B and the point $(1, 0)$. We could then *define* the hyperbolic trig functions to be the coordinates (x, y) of B, that is

$$\cosh\beta = x \tag{14}$$

$$\sinh\beta = y \tag{15}$$

and a little work shows that this definition is exactly the same as the one above. Compare Figure 1. [2]

1.3 Rotations

By analogy with the Euclidean case, we *define* a hyperbolic rotation through the relations

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \tag{16}$$

This corresponds to "rotating" both the x and y axes into the first quadrant, as shown in Figure 1. While this may seem peculiar, it is easily verified that the "distance" is invariant, that is,

$$x^2 - y^2 \equiv x'^2 - y'^2 \tag{17}$$

which follows immediately from the hyperbolic trig identity (4).

[1] No, we haven't defined this. In Euclidean geometry, the the length of a curve is obtained by integrating ds along the curve, where $ds^2 = dx^2 + dy^2$. In a similar way, the Lorentzian length is obtained by integrating $d\sigma$, where $d\sigma^2 = dx^2 - c^2 dt^2$.

[2] As shown in Figure 1, there is another "unit" hyperbola, given by $x^2 - y^2 = -1$. By symmetry, the point A on this hyperbola has coordinates $(x, y) = (\sinh\beta, \cosh\beta)$.

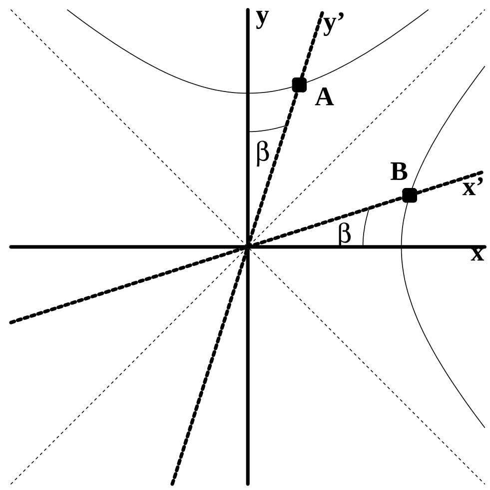

Figure 1: The unit hyperbola. The point A has coordinates $(\sinh\beta, \cosh\beta)$, and $B = (\cosh\beta, \sinh\beta)$.

1.4 Projections

We can ask the same question as we did for Euclidean geometry. Consider a rectangle of width 1 whose sides are parallel to the unprimed axes. How wide is it when measured in the primed coordinates? It turns out that the width of the box in the primed coordinate system is *less than* 1. This is length contraction, to which we will return in the next section, along with time dilation.

1.5 Addition Formulas

What is the slope of the line from the origin to the point A in Figure 1? The equation of this line, the y'-axis, is

$$x = y \tanh\beta \tag{18}$$

Consider now a line with equation

$$x' = y' \tanh\alpha \tag{19}$$

What is its (unprimed) slope? Again, slopes don't add, but (hyperbolic) angles do; the answer is that

$$x = y \tanh(\alpha + \beta) \tag{20}$$

which can be expressed in terms of the slopes $\tanh \alpha$ and $\tanh \beta$ using (7). As discussed in more detail in the next section, this is the Einstein addition formula!

2 The Geometry of Special Relativity

In which it is shown that special relativity is just hyperbolic geometry.

2.1 Spacetime Diagrams

A brilliant aid in understanding special relativity is the *Surveyor's parable* introduced by Taylor and Wheeler [1, 2]. Suppose a town has daytime surveyors, who determine North and East with a compass, nighttime surveyors, who use the North Star. These notions of course differ, since magnetic north is not the direction to the North Pole. Suppose further that both groups measure north/south distances in miles and east/west distances in meters, with both being measured from the town center. How does one go about comparing the measurements of the two groups?

With our knowledge of Euclidean geometry, we see how to do this: Convert miles to meters (or vice versa). Furthermore, distances computed with the Pythagorean theorem do not depend on which group does the surveying. Finally, it is easily seen that "daytime coordinates" can be obtained from "nighttime coordinates" by a simple rotation. The moral of this parable is therefore:

1. *Use the same units.*
2. *The (squared) distance is invariant.*
3. *Different frames are related by rotations.*

Applying that lesson to relativity, the first thing to do is to measure both time and space in the same units. How does one measure distance in seconds? that's easy: simply multiply by c. Thus, since $c = 3 \times 10^8 \ \frac{\text{m}}{\text{s}}$, 1 second of distance is just 3×10^8 m. [3] Note that this has the effect of setting $c = 1$, since the number of seconds (of distance) traveled by light in 1 second (of time) is precisely 1.

Of course, it is also possible to measure time in meters: simply divide by c. Thus, 1 meter of time is the time it takes for light (in vacuum) to travel 1 meter. Again, this has the effect of setting $c = 1$.

[3] A similar unit of distance is the *lightyear*, namely the distance traveled by light in 1 year, which would here be called simply a *year* of distance.

2.2 Lorentz Transformations

The Lorentz transformation between a frame (x,t) at rest and a frame (x',t') moving to the right at speed v was derived in class.The transformation from the moving frame to the frame at rest is given by

$$x = \gamma\,(x' + vt') \tag{21}$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right) \tag{22}$$

where

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{23}$$

The key to converting this to hyperbola geometry is to measure space and time in the same units by replacing t by ct. The transformation from the moving frame, which we now denote (x',ct'), to the frame at rest, now denoted (x,ct), is given by

$$x = \gamma\,(x' + \frac{v}{c}\,ct') \tag{24}$$
$$ct = \gamma\left(ct' + \frac{v}{c}\,x'\right) \tag{25}$$

which makes the symmetry between these equations much more obvious.

We can simplify things still further. Introduce the *rapidity* β via [4]

$$\frac{v}{c} = \tanh\beta \tag{26}$$

Inserting this into the expression for γ we obtain

$$\gamma = \frac{1}{\sqrt{1-\tanh^2\beta}} = \sqrt{\frac{\cosh^2\beta}{\cosh^2\beta - \sinh^2\beta}} = \cosh\beta \tag{27}$$

and

$$\frac{v}{c}\gamma = \tanh\beta\cosh\beta = \sinh\beta \tag{28}$$

Inserting these identities into the Lorentz transformations above brings them to the remarkably simple form

$$x = x'\cosh\beta + ct'\sinh\beta \tag{29}$$
$$ct = x'\sinh\beta + ct'\cosh\beta \tag{30}$$

[4] WARNING: Some authors use β for $\frac{v}{c}$, not the rapidity.

which in matrix form are just

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{31}$$

But (31) is just (16), with $y = ct$!

Thus, Lorentz transformations are just hyperbolic rotations! As noted in the previous section, the invariance of the interval follows immediately from the fundamental hyperbolic trig identity (4). This invariance now takes the form

$$x^2 - c^2t^2 \equiv x'^2 - c^2t'^2 \tag{32}$$

2.3 Addition of Velocities

What is the rapidity β? Consider an observer moving at speed v to the right. This observer's world line intersects the unit hyperbola

$$c^2t^2 - x^2 = 1 \qquad (ct > 0) \tag{33}$$

at the point $A = (\sinh\beta, \cosh\beta)$; this line has "slope" [5]

$$\frac{v}{c} = \tanh\beta \tag{34}$$

as required. Thus, β can be thought of as the *hyperbolic angle* between the ct-axis and the worldline of a moving object. As discussed in the preceding section, β turns out to be precisely the distance from the axis as measured along the hyperbola (in hyperbola geometry!). This is illustrated in Figure 2.

Consider therefore an object moving at speed u relative to an observer moving at speed v. Their rapidities are given by

$$\frac{u}{c} = \tanh\alpha \tag{35}$$

$$\frac{v}{c} = \tanh\beta \tag{36}$$

To determine the resulting speed with respect to an observer at rest, simply add the *rapidities*! One way to think of this is that you are adding the arc

[5] It is not obvious whether "slope" should be defined by $\frac{\Delta x}{c\,\Delta t}$ or by the reciprocal of this expression. This is further complicated by the fact that both (x, ct) and (ct, x) are commonly used to denote the coordinates of the point A!

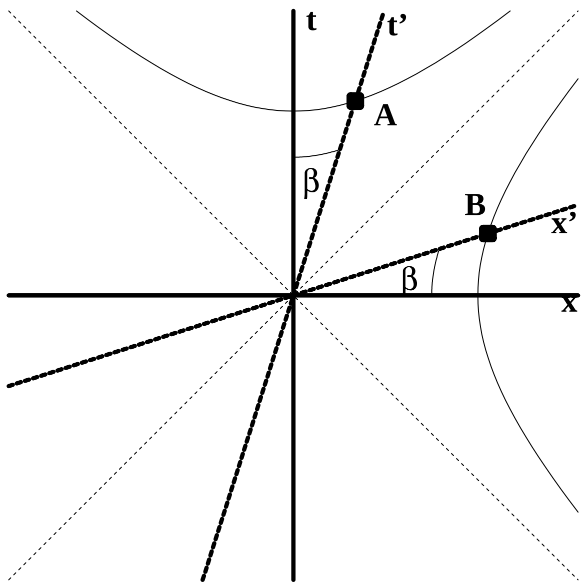

Figure 2: The Lorentz transformation between an observer at rest and an observer moving at speed $\frac{v}{c} = c \tanh \beta$ is shown as a hyperbolic rotation. The point A has coordinates $(\sinh \beta, \cosh \beta)$, and $B = (\cosh \beta, \sinh \beta)$. (Units have been chosen such that $c = 1$.)

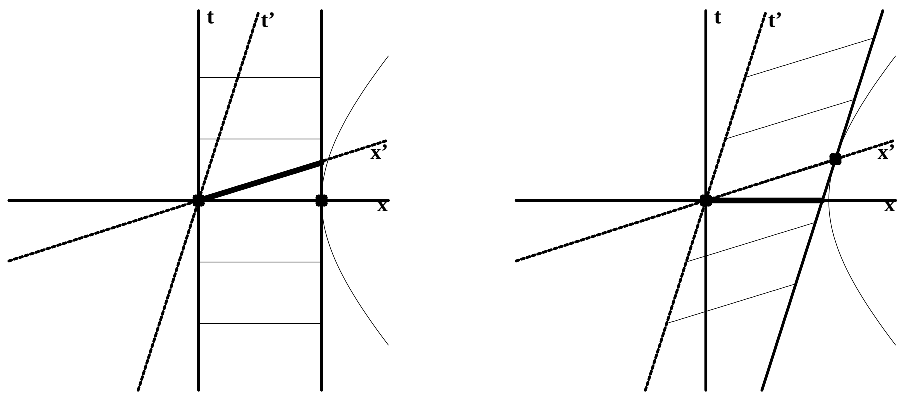

Figure 3: Length contraction as a hyperbolic projection.

lengths along the hyperbola. Another is that you are following a (hyperbolic) rotation through a (hyperbolic) angle β (to get to the moving observer's frame) with a rotation through an angle α. In any case, the resulting speed w is given by

$$\frac{w}{c} = \tanh(\alpha + \beta) = \frac{\tanh\alpha + \tanh\beta}{1 + \tanh\alpha\tanh\beta} = \frac{\frac{u}{c} + \frac{v}{c}}{1 + \frac{uv}{c^2}} \tag{37}$$

which is — finally — precisely the Einstein addition formula!

2.4 Length Contraction

We now return to the question of how "wide" things are.

Consider first a meter stick at rest. In spacetime, the stick "moves" vertically, that is, it ages. This situation is shown in the first sketch in Figure 3, where the horizontal lines show the meter stick at various times (according to an observer at rest). How "wide" is the *worldsheet* of the stick? The observer at rest of course measures the length of the stick by locating both ends *at the same time*, and measuring the distance between them. At $t = 0$, this corresponds to the 2 heavy dots in the sketch, one at the origin and the other on the unit hyperbola. But *all* points on the unit hyperbola are at an interval of 1 meter from the origin. The observer at rest therefore concludes, unsurprisingly, that the meter stick is 1 meter long.

How long does a moving observer think the stick is? This is just the "width" of the worldsheet *as measured by the moving observer*. This observer follows the same procedure, by locating both ends of the stick *at the same time*, and measuring the distance between them. But time now corresponds to t', not t. At $t' = 0$, this measurement corresponds to the heavy line in the sketch. Since this line fails to reach the unit hyperbola, it is clear that the moving observer measures the length of a stationary meter stick to be less than 1 meter. This is length contraction.

To determine the exact value measured by the moving observer, compute the intersection of the line $x = 1$ (the right-hand edge of the meter stick) with the line $t' = 0$ (the x'-axis), or equivalently $ct = x \tanh\beta$, to find that

$$ct = \tanh\beta \tag{38}$$

so that x' is just the interval from this point to the origin, which is

$$x' = \sqrt{x^2 - c^2t^2} = \sqrt{1 - \tanh^2\beta} = \frac{1}{\cosh\beta} \tag{39}$$

What if the stick is moving and the observer is at rest? This situation is shown in the second sketch in Figure 3. The worldsheet now corresponds to a "rotated rectangle", indicated by the parallelograms in the sketch. The fact that the meter stick is 1 meter long in the moving frame is shown by the distance between the 2 heavy dots (along $t' = 0$), and the measurement by the observer at rest is indicated by the heavy line (along $t = 0$). Again, it is clear that the stick appears to have shrunk, since the heavy line fails to reach the unit hyperbola.

Thus, a moving object appears shorter by a factor $1/\cosh\beta$. It doesn't matter whether the stick is moving, or the observer; all that matters is their relative motion.

2.5 Time Dilation

We now investigate moving clocks. Consider first the smaller dot in Figure 4. This corresponds to $ct = 1$ (and $x = 0$), as evidenced by the fact that this point is on the (other) unit hyperbola, as shown. Similarly, the larger dot, lying on the same hyperbola, corresponds to $ct' = 1$ (and $x' = 0$). The horizontal line emanating from this dot gives the value of ct there, which is clearly greater than 1. This is the time measured by the observer at rest when the moving clock says 1; the moving clock therefore runs slow. But now consider the diagonal line emanating from the larger dot. At all points along

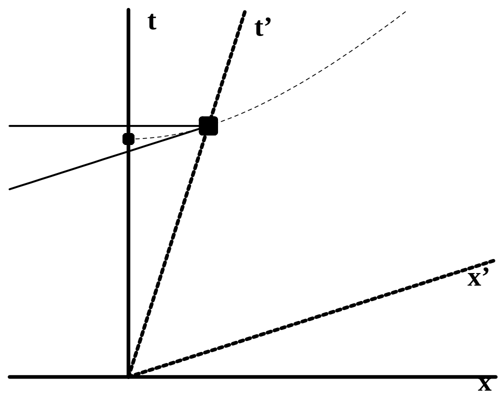

Figure 4: Time dilation as a hyperbolic projection.

this line, $ct' = 1$. In particular, at the smaller dot we must have $ct' > 1$. This is the time measured by the moving observer when the clock at rest says 1; the moving observer therefore concludes the clock at rest runs slow!

There is no contradiction here; one must simply be careful to ask the right question. In each case, observing a clock in another frame of reference corresponds to a projection. In each case, a clock in relative motion to the observer appears to run slow.

2.6 Doppler Shift

The frequency f of a beam of light is related to its wavelength λ by the formula

$$f\lambda = c \tag{40}$$

How do these quantities depend on the observer?

Consider an inertial observer moving to the right in the laboratory frame who is carrying a flashlight which is pointing to the left; see Figure 5. Then the moving observer is traveling along a path of the form $x' = x'_1 = \text{const}$. Suppose the moving observer turns on the flashlight (at time t'_1) just long enough to emit 1 complete wavelength of light, and that this takes time dt'. Then the moving observer "sees" a wavelength

$$\lambda' = c\,dt' \tag{41}$$

According to the lab, the flashlight was turned on at the event (t_1, x_1), and turned off dt_1 seconds later, during which time the moving observer

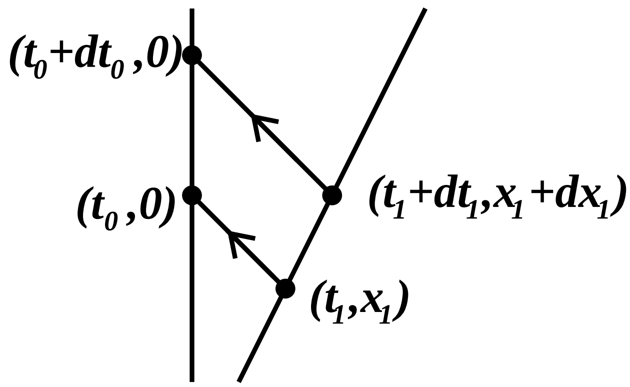

Figure 5: The Doppler effect: An observer moving to the right emits a pulse of light to the left, which is later seen by a stationary observer. The wavelengths measured by the two observers differ, causing a *Doppler shift* in the frequency.

moved a distance dx_1 meters to the right. But when was the light received, at $x = 0$, say?

Let $(t_0, 0)$ denote the first reception of light by a lab observer at $x = 0$, and suppose this observer sees the light stay on for dt_0 seconds. Since light travels at the speed of light, we have the equations

$$\begin{aligned} c(t_0 - t_1) &= x_1 && (42) \\ c[(t_0 + dt_0) - (t_1 + dt_1)] &= x_1 + dx_1 && (43) \end{aligned}$$

from which it follows that

$$dt_0 - dt_1 = dx_1 \qquad (44)$$

so that

$$\begin{aligned} dt_0 &= dx_1 + dt_1 && (45) \\ &= (dx_1' \cosh\beta + dt_1' \sinh\beta) + (dt_1' \cosh\beta + dx_1' \sinh\beta) && (46) \\ &= \cosh\beta + \sinh\beta dt_1' && (47) \end{aligned}$$

But the wavelength as seen in the lab is

$$\lambda = c\, dt_0 \qquad (48)$$

so that

$$\frac{\lambda}{\lambda'} = \frac{dt_0}{dt'_1} = \cosh\beta + \sinh\beta \tag{49}$$

$$= \cosh\beta\,(1+\tanh\beta) = \gamma\,(1+\frac{v}{c}) = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tag{50}$$

The frequencies transform inversely, that is

$$\frac{f'}{f} = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \tag{51}$$

References

[1] Edwin F. Taylor and John Archibald Wheeler, **Spacetime Physics**, W. H. Freeman, San Francisco, 1963.

[2] Edwin F. Taylor and John Archibald Wheeler, **Spacetime Physics**, second edition, W. H. Freeman, New York, 1992.