

4 Relativistic Electromagnetism

In which it is shown that electricity and magnetism can no more be separated than space and time.

4.1 Magnetism from Electricity

Our starting point is the electric and magnetic fields of an infinite straight wire, which are derived in most introductory textbooks on electrodynamics, such as Griffiths [3], and which we state here without proof.

The electric field of an infinite straight wire with charge density λ points away from the wire with magnitude

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \quad (96)$$

where r is the perpendicular distance from the wire and ϵ_0 is the permittivity constant. The magnetic field of such a wire with current density I has magnitude

$$B = \frac{\mu_0 I}{2\pi r} \quad (97)$$

with r as above, and where μ_0 is the permeability constant, which is related to ϵ_0 by

$$\epsilon_0 \mu_0 = \frac{1}{c^2} \quad (98)$$

(The direction of the magnetic field is obtained as the cross product of the direction of the current and the position vector from the wire to the point in question.)

We will also need the *Lorentz force law*, which says that the force \vec{F} on a test particle of charge q and velocity \vec{v} is given by

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (99)$$

where \vec{E} and \vec{B} denote the electric and magnetic fields (with magnitudes E and B , respectively).

Consider an infinite line charge, consisting of identical particles of charge ρ , separated by a distance ℓ . This gives an infinite wire with (average) charge density

$$\lambda_0 = \frac{\rho}{\ell} \quad (100)$$

Suppose now that the charges are moving to the right with speed

$$u = c \tanh \alpha \quad (101)$$

Due to length contraction, the charge density seen by an observer at rest *increases* to

$$\lambda = \frac{\rho}{\frac{\ell}{\cosh \alpha}} = \lambda_0 \cosh \alpha \quad (102)$$

Suppose now that there are positively charged particles moving to the right, and equally but negatively charged particles moving to the left, each with speed u . Consider further a test particle of charge q situated a distance r from the wire and moving with speed

$$v = c \tanh \beta \quad (103)$$

to the right. Then the net charge density in the laboratory frame is 0, so that there is no electrical force on the test particle in this frame. There is of course a net current density, however, namely

$$I = \lambda u + (-\lambda)(-u) = 2\lambda u \quad (104)$$

What does the test particle see? Switch to the rest frame of the test particle; this makes the negative charges appear to move faster, with speed $u_- > u$, and the positive charges move slower, with speed $u_+ < u$. The relative speeds satisfy

$$\frac{u_+}{c} = \tanh(\alpha - \beta) \quad (105)$$

$$\frac{u_-}{c} = \tanh(\alpha + \beta) \quad (106)$$

resulting in current densities

$$\lambda_{\pm} = \lambda \cosh(\alpha \mp \beta) = \lambda(\cosh \alpha \cosh \beta \mp \sinh \alpha \sinh \beta) \quad (107)$$

resulting in a total charge density of

$$\lambda' = \lambda_+ - \lambda_- \quad (108)$$

$$= -2\lambda_0 \sinh \alpha \sinh \beta \quad (109)$$

$$= -2\lambda \tanh \alpha \sinh \beta \quad (110)$$

According to (96), this results in an electric field of magnitude

$$E' = \frac{\lambda'}{2\pi\epsilon} \quad (111)$$

which in turn leads to an electric force of magnitude

$$F' = qE' = -\frac{\lambda}{\pi\epsilon_0 r} q \tanh \alpha \sinh \beta \quad (112)$$

$$= -\frac{\lambda u}{\pi\epsilon_0 c^2 r} qv \cosh \beta \quad (113)$$

$$= -\frac{\mu_0 I}{2\pi r} qv \cosh \beta \quad (114)$$

To relate this to the force observed in the laboratory frame, we must consider how force transforms under a Lorentz transformation. We have ⁶

$$\vec{F}' = \frac{d\vec{p}'}{dt'} \quad (115)$$

and of course also

$$\vec{F} = \frac{d\vec{p}}{dt} \quad (116)$$

But since in this case the force is perpendicular to the direction of motion, we have

$$d\vec{p} = d\vec{p}' \quad (117)$$

and since $dx' = 0$ in the comoving frame we also have

$$dt = dt' \cosh \beta \quad (118)$$

Thus, in this case, the magnitudes are related by

$$F = \frac{F'}{\cosh \beta} = -\frac{\mu_0 I}{2\pi r} qv \quad (119)$$

But this is just the Lorentz force law

$$\vec{F} = q\vec{v} \times \vec{B} \quad (120)$$

with $B = |\vec{B}|$ given by (97)!

We conclude that in the laboratory frame there is a *magnetic* force on the test particle, which is just the *electric* force observed in the comoving frame!

⁶This is the traditional notion of force, which does not transform simply between frames. As discussed briefly below, a possibly more useful notion of force is obtained by differentiating with respect to proper time.

4.2 Lorentz Transformations

We now investigate more general transformations of electric and magnetic fields between different inertial frames. Our starting point is the electromagnetic field of an infinite flat metal sheet, which is derived in most introductory textbooks on electrodynamics, such as Griffiths [3], and which we state here without proof.

The electric field of an infinite metal sheet with charge density σ points away from the sheet and has the constant magnitude

$$|E| = \frac{\sigma}{2\epsilon_0} \quad (121)$$

The magnetic field of such a sheet with current density $\vec{\kappa}$ has constant magnitude

$$|B| = \frac{\mu}{2} |\vec{\kappa}| \quad (122)$$

and direction determined by the right-hand-rule.

Consider a capacitor consisting of 2 horizontal ($z = \text{constant}$) parallel plates, with equal and opposite charge densities. For definiteness, take the charge density on the bottom plate to be σ_0 , and suppose that the charges are at rest, that is, that the current density of each plate is zero. Then the electric field is given by

$$\vec{E}_0 = E_0 \vec{j} = \frac{\sigma_0}{\epsilon_0} \vec{j} \quad (123)$$

between the plates and vanishes elsewhere. Now let the capacitor move to the left with velocity

$$\vec{u} = -u \vec{i} = -c \tanh \alpha \vec{i} \quad (124)$$

Then the *width* of the plate is unchanged, but, just as for the line charge (102), the *length* is Lorentz contracted, which *decreases* the area, and hence *increases* the charge density. The charge density (on the bottom plate) is therefore

$$\sigma = \sigma_0 \cosh \alpha \quad (125)$$

But there is now also a current density, which is given by

$$\vec{\kappa} = \sigma \vec{u} \quad (126)$$

on the lower plate. The top plate has charge density $-\sigma$, so its current density is $-\vec{\kappa}$. Then both the electric and magnetic fields vanish outside the

plates, whereas inside the plates one has

$$\vec{\mathbf{E}} = E^y \vec{\mathbf{j}} = \frac{\sigma}{\epsilon_0} \vec{\mathbf{j}} \quad (127)$$

$$\vec{\mathbf{B}} = B^z \vec{\mathbf{k}} = -\mu_0 \sigma u \vec{\mathbf{k}} \quad (128)$$

which can be rewritten using (124) and (125) in the form

$$E^y = E_0 \cosh \alpha \quad (129)$$

$$B^z = cB_0 \sinh \alpha \quad (130)$$

For later convenience, we have introduced in the last equation the quantity

$$B_0 = -\mu_0 \sigma_0 = -\mu_0 \epsilon_0 E_0 = -\frac{1}{c^2} E_0 \quad (131)$$

which does *not* correspond to the magnetic field when the plate is at rest — which of course vanishes since $\vec{\mathbf{u}} = 0$.

The above discussion gives the electric and magnetic fields seen by an observer at rest. What is seen by an observer moving to the right with speed $v = c \tanh \beta$? To compute this, first use the velocity addition law to compute the correct rapidity to insert in (130), which is simply the sum of the rapidities α and β !

The moving observer therefore sees an electric field $\vec{\mathbf{E}}'$ and a magnetic field $\vec{\mathbf{B}}'$. From (130), (131), and the hyperbolic trig formulas (5) and (6), we have

$$\begin{aligned} E'^y &= E_0 \cosh(\alpha + \beta) \\ &= E_0 \cosh \alpha \cosh \beta + E_0 \sinh \alpha \sinh \beta \\ &= E_0 \cosh \alpha \cosh \beta - c^2 B_0 \sinh \alpha \sinh \beta \\ &= E^y \cosh \beta - cB^z \sinh \beta \end{aligned} \quad (132)$$

and similarly

$$\begin{aligned} B'^z &= cB_0 \sinh(\alpha + \beta) \\ &= cB_0 \sinh \alpha \cosh \beta + cB_0 \cosh \alpha \sinh \beta \\ &= B^z \cosh \beta - \frac{1}{c} E^y \sinh \beta \end{aligned} \quad (133)$$

Repeating the argument with the y and z axes interchanged (and being careful about the orientation), we obtain the analogous formulas

$$E'^z = E^z \cosh \beta + cB^y \sinh \beta \quad (134)$$

$$B'^x = B^y \cosh \beta + \frac{1}{c} E^z \sinh \beta \quad (135)$$

Finally, by considering motion perpendicular to the plates one can show [3]

$$E'^x = E^x \quad (136)$$

and by considering a solenoid one obtains [3]

$$B'^x = B^x \quad (137)$$

Equations (132)–(137) describe the behavior of the electric and magnetic fields under Lorentz transformations. These equations can be nicely rewritten in vector language by introducing the projections parallel and perpendicular to the direction of motion of the observer, namely

$$\vec{E}_{\parallel} = \frac{\vec{v} \cdot \vec{E}}{\vec{v} \cdot \vec{v}} \vec{v} \quad (138)$$

$$\vec{B}_{\parallel} = \frac{\vec{v} \cdot \vec{B}}{\vec{v} \cdot \vec{v}} \vec{v} \quad (139)$$

and

$$\vec{E}_{\perp} = \vec{E} - \vec{E}_{\parallel} \quad (140)$$

$$\vec{B}_{\perp} = \vec{B} - \vec{B}_{\parallel} \quad (141)$$

We then have

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \quad (142)$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \quad (143)$$

and

$$\vec{B}'_{\perp} = \left(\vec{B}_{\perp} - \frac{1}{c^2} \vec{v} \times \vec{E}_{\perp} \right) \cosh \beta \quad (144)$$

$$\vec{E}'_{\perp} = \left(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp} \right) \cosh \beta \quad (145)$$

4.3 Vectors

In the previous section, we used 2-component vectors to describe spacetime, with one component for time and the other for space. In the case of 3 spatial dimensions, we use 4-component vectors, namely

$$x^\nu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (146)$$

These are called *contravariant* vectors, and their indices are written “upstairs”, that is, as superscripts.

Just as before, Lorentz transformations are hyperbolic rotations, which must now be written as 4×4 matrices. For instance, a “boost” in the x direction now takes the form

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh \beta & -\sinh \beta & 0 & 0 \\ -\sinh \beta & \cosh \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad (147)$$

A general Lorentz transformation can be written in the form

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (148)$$

where $\Lambda^\mu{}_\nu$ are (the components of) the appropriate 4×4 matrix, and where we have adopted the *Einstein summation convention* that repeated indices, in this case ν , are to be summed from 0 to 3. In matrix notation, this can be written as

$$\mathbf{x}' = \mathbf{\Lambda} \mathbf{x} \quad (149)$$

Why are some indices up and others down? In relativity, both special and general, it is essential to distinguish between 2 types of vectors. In addition to contravariant vectors, there are also *covariant* vectors, often referred to as dual vectors. The dual vector associated with x^μ is ⁷

$$x_\mu = \begin{pmatrix} -x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -ct \\ x \\ y \\ z \end{pmatrix} \quad (150)$$

⁷Some authors use different conventions.

We won't have much need for covariant vectors, but note that the invariance of the interval can be nicely written as

$$\begin{aligned} x_\mu x^\mu &= -c^2 t^2 + x^2 + y^2 + z^2 \\ &= x'^\mu x'_\mu \end{aligned} \quad (151)$$

(Don't forget the summation convention!) In fact, this equation can be taken as the *definition* of Lorentz transformations, and it is straightforward to determine which matrices $\Lambda^\mu{}_\nu$ are allowed.

Taking the derivative with respect to proper time leads to the *4-velocity*

$$u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt} \frac{dt}{d\tau} \quad (152)$$

It is often useful to divide these into space and time in the form

$$u = \begin{pmatrix} c\gamma \\ \vec{v}\gamma \end{pmatrix} = \begin{pmatrix} c \cosh \beta \\ \hat{v} c \sinh \beta \end{pmatrix} \quad (153)$$

where \hat{v} is the unit vector in the direction of \vec{v} . Note that the 4-velocity is a unit vector in the sense that

$$\frac{1}{c^2} u_\mu u^\mu = -1 \quad (154)$$

The 4-momentum is simply the 4-velocity times the rest mass, that is

$$p^\mu = m u^\mu = \begin{pmatrix} \frac{1}{c} E \\ \vec{p} \end{pmatrix} = \begin{pmatrix} mc\gamma \\ m\vec{v}\gamma \end{pmatrix} = \begin{pmatrix} mc \cosh \beta \\ \hat{v} mc \sinh \beta \end{pmatrix} \quad (155)$$

and note that

$$p_\mu p^\mu = -m^2 c^2 \quad (156)$$

which is equivalent to our earlier result

$$E^2 - p^2 c^2 = m^2 c^4 \quad (157)$$

4.4 Tensors

Roughly speaking, tensors are like vectors, but with more components, and hence more indices. We will only consider one particular case, namely *rank 2 contravariant tensors*, which have 2 “upstairs” indices. Such a tensor has

components in a particular reference frame which make up a 4×4 matrix, namely

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix} \quad (158)$$

How does \mathbf{T} transform under Lorentz transformations? Well, it has *two* indices, *each* of which must be transformed. This leads to a transformation of the form

$$T'^{\mu\nu} = \Lambda^\mu_\rho \Lambda^\nu_\sigma T^{\rho\sigma} = \Lambda^\mu_\rho T^{\rho\sigma} \Lambda^\nu_\sigma \quad (159)$$

where the second form (and the summation convention!) leads naturally to the matrix equation

$$\mathbf{T}' = \mathbf{\Lambda T \Lambda}^t \quad (160)$$

where t denotes matrix transpose.

Further simplification occurs in the special case where \mathbf{T} is antisymmetric, that is

$$T^{\nu\mu} = -T^{\mu\nu} \quad (161)$$

so that the components of \mathbf{T} take the form

$$T^{\mu\nu} = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & f & -e \\ -b & -f & 0 & d \\ -c & e & -d & 0 \end{pmatrix} \quad (162)$$

4.5 The Electromagnetic Field

Why have we done all this? Well, first of all, note that, due to antisymmetry, \mathbf{T} has precisely 6 independent components. Next, compute \mathbf{T}' , using matrix multiplication and the fundamental hyperbolic trig identity (4). As you should check for yourself, the result is

$$T'^{\mu\nu} = \begin{pmatrix} 0 & a' & b' & c' \\ -a' & 0 & f' & -e' \\ -b' & -f' & 0 & d' \\ -c' & e' & -d' & 0 \end{pmatrix} \quad (163)$$

where

$$a' = a \quad (164)$$

$$b' = b \cosh \beta - f \sinh \beta \quad (165)$$

$$c' = c \cosh \beta + e \sinh \beta \quad (166)$$

$$d' = d \quad (167)$$

$$e' = e \cosh \beta + c \sinh \beta \quad (168)$$

$$f' = f \cosh \beta - b \sinh \beta \quad (169)$$

The first 3 of these are the transformation rule for the electric field, and the remaining 3 are the transformation rule for the magnetic field!

We are thus led to introduce the *electromagnetic field tensor*, namely

$$F^{uv} = \begin{pmatrix} 0 & \frac{1}{c} E^x & \frac{1}{c} E^y & \frac{1}{c} E^z \\ -\frac{1}{c} E^x & 0 & B^z & -B^y \\ -\frac{1}{c} E^y & -B^z & 0 & B^x \\ -\frac{1}{c} E^z & B^y & -B^x & 0 \end{pmatrix} \quad (170)$$

4.6 Maxwell's equations

Maxwell's equations in vacuum (and in MKS units) are

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \quad (171)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (172)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (173)$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (174)$$

where ρ is the charge density, \vec{J} is the current density, and the constants μ_0 and ϵ_0 satisfy (98). Equation (171) is just Gauss' Law, (173) is Faraday's equation, and (174) is Ampère's Law corrected for the case of a time-dependent electric field. We also have the charge conservation equation

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad (175)$$

and the Lorentz force law

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (176)$$

The middle two of Maxwell equations are automatically solved by introducing the scalar potential Φ and the vector potential \vec{A} and defining

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (177)$$

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \Phi \quad (178)$$

4.7 Tensor Version of Maxwell's Equations

Consider the following derivatives of F :

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \frac{\partial F^{\mu 0}}{\partial t} + \frac{\partial F^{\mu 1}}{\partial x} + \frac{\partial F^{\mu 2}}{\partial y} + \frac{\partial F^{\mu 3}}{\partial z} \quad (179)$$

This corresponds to four different expressions, one for each value of μ . For $\mu = 0$, we get

$$0 + \frac{1}{c} \frac{\partial E^x}{\partial x} + \frac{1}{c} \frac{\partial E^y}{\partial y} + \frac{1}{c} \frac{\partial E^z}{\partial z} = \frac{1}{c} \vec{\nabla} \cdot \vec{E} = \frac{\rho}{c\epsilon_0} = c\mu_0\rho \quad (180)$$

where Gauss' Law was used to get the final two equalities. Similarly, for $\mu = 1$ we have

$$-\frac{1}{c^2} \frac{\partial E^x}{\partial t} + 0 + \frac{\partial B^z}{\partial y} - \frac{\partial B^y}{\partial z} \quad (181)$$

and combining this with the expressions for $\mu = 2$ and $\mu = 3$ yields the left-hand-side of

$$-\frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \nabla \times \vec{B} = \mu_0 \vec{J} \quad (182)$$

where the right-hand-side follows from Ampère's Law. Combining these equations, and introducing the *4-current density*

$$J^\mu = \begin{pmatrix} c\rho \\ \vec{J} \end{pmatrix} \quad (183)$$

leads to

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 \vec{J}^\mu \quad (184)$$

which is equivalent to the two Maxwell equations with a physical source, namely Gauss' Law and Ampère's Law.

Furthermore, taking the (4-dimensional!) divergence of the 4-current density leads to

$$\mu_0 \frac{\partial J^\mu}{\partial x^\mu} = \frac{\partial}{\partial x^\mu} \frac{\partial F^{\mu\nu}}{\partial x^\nu} = 0 \quad (185)$$

since there is an implicit double sum over both μ and ν , and the derivatives commute but $F^{\mu\nu}$ is antisymmetric. (Check this by interchanging the order of summation.) Working out the components of this equation, we have

$$\frac{1}{c} \frac{\partial J^0}{\partial t} + \frac{\partial J^1}{\partial x} + \frac{\partial J^2}{\partial y} + \frac{\partial J^3}{\partial z} = 0 \quad (186)$$

which is just the charge conservation equation (175).

What about the remaining equations? Introduce the *dual tensor* $G^{\mu\nu}$ obtained from $F^{\mu\nu}$ by replacing $\frac{1}{c}\vec{E}$ by \vec{B} and \vec{B} by $-\frac{1}{c}\vec{E}$, resulting in

$$G^{\mu\nu} = \begin{pmatrix} 0 & B^x & B^y & B^z \\ -B^x & 0 & -\frac{1}{c}E^z & \frac{1}{c}E^y \\ -B^y & \frac{1}{c}E^z & 0 & -\frac{1}{c}E^x \\ -B^z & -\frac{1}{c}E^y & \frac{1}{c}E^x & 0 \end{pmatrix} \quad (187)$$

Then the four equations

$$\frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0 \quad (188)$$

correspond to

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (189)$$

$$-\frac{1}{c} \frac{\partial \vec{B}}{\partial t} - \frac{1}{c} \vec{\nabla} \times \vec{E} = 0 \quad (190)$$

which are precisely the two remaining Maxwell equations.

Some further properties of these tensors are

$$\frac{1}{2} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{c^2} |\vec{E}|^2 + |\vec{B}|^2 = -\frac{1}{2} G_{\mu\nu} G^{\mu\nu} \quad (191)$$

$$\frac{1}{4} G_{\mu\nu} F^{\mu\nu} = -\frac{1}{c} \vec{E} \cdot \vec{B} \quad (192)$$

where care must be taken with the signs of the components of the *covariant* tensors $F_{\mu\nu}$ and $G_{\mu\nu}$. You may recognize these equations as corresponding to important scalar invariants of the electromagnetic field.

Finally, it is possible to write solve the sourcefree Maxwell equations by introducing a *4-potential*

$$A^\mu = \begin{pmatrix} \frac{1}{c} \Phi \\ \vec{\mathbf{A}} \end{pmatrix} \quad (193)$$

and defining

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu} \quad (194)$$

where again care must be taken with the signs of the components with “downstairs” indices. Furthermore, the Lorentz force law can be rewritten in the form

$$m \frac{\partial p^\mu}{\partial \tau} = q u_\nu F^{\mu\nu} \quad (195)$$

Note the appearance of the proper time τ in this equation. Just as in the previous section, this is because differentiation with respect to τ pulls through a Lorentz transformation, which makes this a valid tensor equation, valid in any inertial frame.

References

- [1] Edwin F. Taylor and John Archibald Wheeler, **Spacetime Physics**, W. H. Freeman, San Francisco, 1963.
- [2] Edwin F. Taylor and John Archibald Wheeler, **Spacetime Physics**, second edition, W. H. Freeman, New York, 1992.
- [3] David J. Griffiths, **Introduction to Electrodynamics**, 3rd edition, Prentice Hall, Upper Saddle River, 1999.