

Capstones in Physics: Electromagnetism

5. WAVES

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5.1. Plane Waves

5.1.A. WAVE EQUATIONS IN UNIFORM MEDIA

FARADAY	$\mathbf{B} = \mu\mathbf{H}, \mathbf{D} = \epsilon\mathbf{E}$	AMPERE
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$$\nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} - \epsilon \frac{\partial \mathbf{E}}{\partial t} = \mathbf{J}_{\text{free}}$$

curl

$$\nabla \times (\nabla \times \mathbf{E}) + \mu \nabla \times \frac{\partial \mathbf{H}}{\partial t} = 0$$

$$\nabla \times (\nabla \times \mathbf{H}) - \epsilon \nabla \times \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{J}_{\text{free}}$$

time derivative, rearrange

$$\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

$$\nabla \times \frac{\partial \mathbf{H}}{\partial t} = \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \frac{\partial}{\partial t} \mathbf{J}_{\text{free}}$$

substitute, rearrange

$$\nabla \times (\nabla \times \mathbf{E}) + \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu \frac{\partial}{\partial t} \mathbf{J}_{\text{free}}$$

$$\nabla \times (\nabla \times \mathbf{H}) + \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = \nabla \times \mathbf{J}_{\text{free}}$$

use identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, rearrange

$$\nabla^2 \mathbf{E} - \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial}{\partial t} \mathbf{J}_{\text{free}} + \nabla(\nabla \cdot \mathbf{E}) \quad \nabla^2 \mathbf{H} - \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla \times \mathbf{J}_{\text{free}} + \nabla(\nabla \cdot \mathbf{H})$$

evaluate divergences

$$\nabla^2 \mathbf{E} - \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu \frac{\partial}{\partial t} \mathbf{J}_{\text{free}} + \frac{\nabla \rho_{\text{free}}}{\epsilon}$$

$$\nabla^2 \mathbf{H} - \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = -\nabla \times \mathbf{J}_{\text{free}}$$

evalutate $\mathbf{J}_{\text{free}} = \sigma \mathbf{E}$, curl from Faraday law

$$\nabla^2 \mathbf{E} - \epsilon \mu \frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma \mu \frac{\partial \mathbf{E}}{\partial t} = \frac{\nabla \rho_{\text{free}}}{\epsilon}$$

$$\nabla^2 \mathbf{H} - \epsilon \mu \frac{\partial^2 \mathbf{H}}{\partial t^2} - \sigma \mu \frac{\partial \mathbf{H}}{\partial t} = 0$$

5.1. B. LINEARLY POLARIZED MONOCHROMATIC PLANE WAVES

WAVE EQUATIONS WITHOUT SOURCES $\rho_{\text{free}} = 0$ and $\mathbf{J}_{\text{free}} = \mathbf{0}$

$$\nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \nabla^2 \mathbf{H} - \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

where $\epsilon\mu = 1/v^2$

In a **linearly polarized monochromatic plane wave**, \mathbf{E} , \mathbf{H} are the real parts of position-dependent complex exponentials given by

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \qquad \mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

substitute in Maxwell eqs:

conclusion

$$\nabla \cdot \mathbf{E} = 0 \rightarrow \mathbf{k} \cdot \mathbf{E} = 0 \qquad \Rightarrow \qquad \mathbf{E} \text{ is transverse, } \mathbf{E} \perp \mathbf{k}$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \mathbf{k} \cdot \mathbf{H} = 0 \qquad \Rightarrow \qquad \mathbf{H} \text{ is transverse, } \mathbf{H} \perp \mathbf{k}$$

$$\nabla \times \mathbf{E} + \mu \frac{\partial \mathbf{H}}{\partial t} = 0 \rightarrow \mathbf{k} \times \mathbf{E} = \omega\mu \mathbf{H} \qquad \Rightarrow \qquad \mathbf{E} \text{ and } \mathbf{H} \text{ are orthogonal, } \mathbf{E} \perp \mathbf{H}$$

$$\nabla \times \mathbf{H} - \epsilon \frac{\partial \mathbf{E}}{\partial t} = 0 \rightarrow \mathbf{k} \times \mathbf{H} = -\epsilon\omega \mathbf{E} \qquad \Rightarrow \qquad \mathbf{E} \text{ and } \mathbf{H} \text{ are orthogonal, } \mathbf{E} \perp \mathbf{H}$$

substitute \mathbf{E} , \mathbf{H} in wave equations: $k^2 = \epsilon\mu \omega^2$, $k(\omega) = \omega \sqrt{\epsilon\mu}$

phase velocity $v_p \equiv \frac{k}{\omega} = \frac{1}{\sqrt{\epsilon\mu}}$, in vacuum $v_p = \frac{1}{\sqrt{\epsilon_0\mu_0}} = c$: speed of light

index of refraction $n = \sqrt{\epsilon_r \mu_r} \approx \sqrt{\epsilon_r}$, e.g., $n \approx 1.5$ for glass

$$\mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E} \Rightarrow B_0 = \frac{k}{\omega} E_0 = E_0 / v_p, \text{ in vacuum } E_0 = cB_0$$

ENERGY AND MOMENTUM IN EM WAVES

Energy per unit volume: $u = \frac{1}{2} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right)$

In the case of plane wave propagating in the z direction

$$B^2 = \epsilon\mu E^2 \Rightarrow u = \epsilon E^2 = \epsilon E_0^2 \cos^2(kz - \omega t + \delta)$$

Energy flux density (Poynting vector):

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \hat{z} v \epsilon E_0^2 \cos^2(kz - \omega t + \delta) = v u \hat{z}$$

Momentum density:

$$\mathbf{P} = \frac{1}{v^2} \mathbf{S} = \hat{z} \frac{1}{v} \epsilon E_0^2 \cos^2(kz - \omega t + \delta) = \frac{1}{v} u \hat{z}$$

Time average over a complete cycle

$$\langle u \rangle = \frac{1}{2} \epsilon E_0^2, \quad \langle \mathbf{S} \rangle = \frac{1}{2} v \epsilon E_0^2 \hat{z}, \quad \text{and} \quad \langle \mathbf{P} \rangle = \frac{1}{2v} \epsilon E_0^2 \hat{z}, \quad \text{since} \quad \langle \cos^2 \rangle = \frac{1}{2}.$$

Intensity of EM wave is defined as $I = \langle S \rangle = \frac{1}{2} v \epsilon E_0^2$.

5.2. Boundaries

5.2.A. REFLECTION AND REFRACTION

Plane boundary between media \Rightarrow

need Maxwell (wave) eqs.

plus boundary conditions

far away from boundary:

Incident wave in medium 1: $\mathbf{E}_I, \mathbf{H}_I, \mathbf{k}_I$

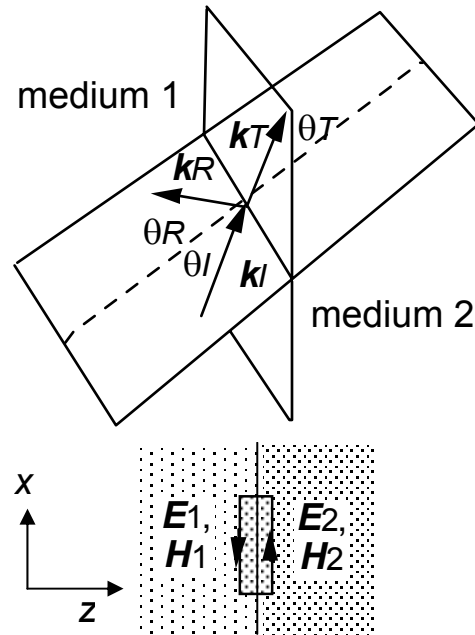
reflected wave in medium 1: $\mathbf{E}_R, \mathbf{H}_R, \mathbf{k}_R$

transmitted wave in medium 2: $\mathbf{E}_T, \mathbf{H}, \mathbf{k}_T$

at the interface between media:

$\mathbf{E} \times \hat{\mathbf{n}}$ continuous (Ampere's law)

$\mathbf{H} \times \hat{\mathbf{n}}$ continuous (Faraday's law)



medium 1

$$k_I = k_R = k_1 = n_1 k_0$$

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}(\mathbf{E}_I + \mathbf{E}_R)$$

$$\mathbf{H}(\mathbf{r}, t) = \text{Re}(\mathbf{H}_I + \mathbf{H}_R)$$

$$\mathbf{E}_I(\mathbf{r}, t) = \mathbf{E}_{I0} e^{i(\mathbf{k}_I \cdot \mathbf{r} - \omega t)}$$

$$\mathbf{E}_R(\mathbf{r}, t) = \mathbf{E}_{R0} e^{i(\mathbf{k}_R \cdot \mathbf{r} - \omega t)}$$

interface

$$k_0 \equiv \omega/c$$

$\mathbf{E} \times \hat{\mathbf{n}}$ continuous

$\mathbf{H} \times \hat{\mathbf{n}}$ continuous

at all times

$$\omega_I = \omega_R = \omega_T = \omega$$

medium 2

$$k_T = k_2 = n_2 k_0$$

$$\mathbf{E}(\mathbf{r}, t) = \text{Re} \mathbf{E}_T$$

$$\mathbf{H}(\mathbf{r}, t) = \text{Re} \mathbf{H}_T$$

$$\mathbf{E}_T(\mathbf{r}, t) = \mathbf{E}_{T0} e^{i(\mathbf{k}_T \cdot \mathbf{r} - \omega t)}$$

match on interface ($z = 0$): $\mathbf{E}_I + \mathbf{E}_R = \mathbf{E}_T, \mathbf{H}_I + \mathbf{H}_R = \mathbf{H}_T$

$$\mathbf{k}_I \cdot \mathbf{r} = \mathbf{k}_R \cdot \mathbf{r} = \mathbf{k}_T \cdot \mathbf{r} \Rightarrow k_I \sin \theta_I = k_R \sin \theta_R = k_T \sin \theta_T$$

i. Plane of incidence: \mathbf{k}_I lies in the xz plane. $\Rightarrow \mathbf{k}_R$ and \mathbf{k}_T in the xz plane.

ii. Law of reflection: $k_I \sin \theta_I = k_R \sin \theta_R, k_I = k_R \Rightarrow \theta_I = \theta_R$

iii. Law of refraction (Snell's law):

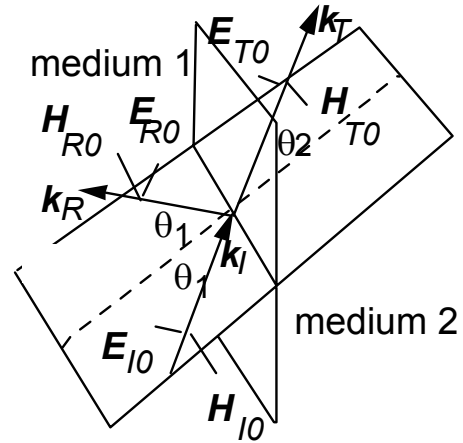
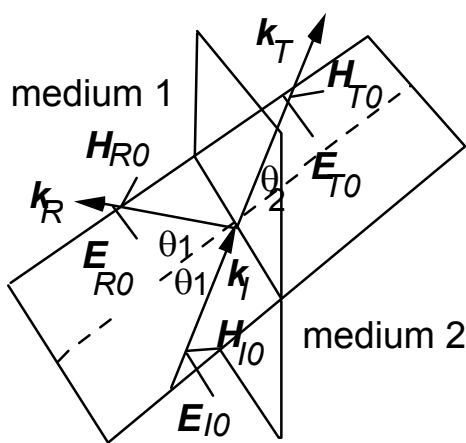
$$k_I \sin \theta_I = k_T \sin \theta_T, k_I / k_T = n_1 / n_2 \Rightarrow \boxed{n_1 \sin \theta_I = n_2 \sin \theta_T}$$

5.2.B. FRESNEL EQUATIONS

Write components with respect to plane of incidence:

$\mathbf{E} = \mathbf{E}_{||} + \mathbf{E}_{\perp}$, $\mathbf{H} = \mathbf{H}_{||} + \mathbf{H}_{\perp}$: $\mathbf{E}_{||}$, $\mathbf{H}_{||}$ in the xz plane and \mathbf{E}_{\perp} , \mathbf{H}_{\perp} in the y direction

Boundary : $\mathbf{E} \times \hat{\mathbf{z}}$, $\mathbf{H} \times \hat{\mathbf{z}}$ continuous; note $\mathbf{H} = \frac{1}{\omega\mu} \mathbf{k} \times \mathbf{E}$



\mathbf{E}_{\perp} , $\mathbf{H}_{||}$ CASES $\mathbf{E}_{||}$, \mathbf{H}_{\perp}

$$E(1) \times \hat{\mathbf{z}} = E(2) \times \hat{\mathbf{z}}$$

$$E_{i0} + E_{R0} = E_{T0}$$

$$E_{i0} \cos \theta_1 + E_{R0} \cos \theta_1 = E_{T0} \cos \theta_2$$

$$H(1) \times \hat{\mathbf{z}} = H(2) \times \hat{\mathbf{z}}$$

$$H_{i0} \cos \theta_1 - H_{R0} \cos \theta_1 = H_{T0} \cos \theta_2$$

$$H_{i0} - H_{R0} = H_{T0}$$

$$Z = E/H = \mu c/n$$

$$(E_{i0} - E_{R0}) \frac{\cos \theta_1}{Z_1} = E_{T0} \frac{\cos \theta_2}{Z_2}$$

$$(E_{i0} - E_{R0})/Z_1 = E_{T0}/Z_2$$

solve

$$E_{T0\perp} = \frac{2 \cos \theta_1 / Z_1}{\cos \theta_1 / Z_1 + \cos \theta_2 / Z_2} E_{i0\perp};$$

$$E_{T0\parallel} = \frac{2 Z_2 \cos \theta_1}{Z_2 \cos \theta_2 + Z_1 \cos \theta_1} E_{i0\parallel}$$

FRESNEL EQUATIONS

$$E_{R0\perp} = \frac{\cos \theta_1 / Z_1 - \cos \theta_2 / Z_2}{\cos \theta_1 / Z_1 + \cos \theta_2 / Z_2} E_{i0\perp};$$

$$E_{R0\parallel} = \frac{Z_2 \cos \theta_2 - Z_1 \cos \theta_1}{Z_2 \cos \theta_2 + Z_1 \cos \theta_1} E_{i0\parallel}$$

Components with respect to plane of incidence ($\mu_1=\mu_2=\mu_0$):

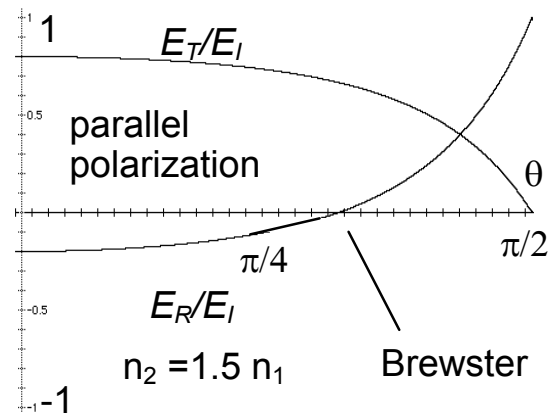
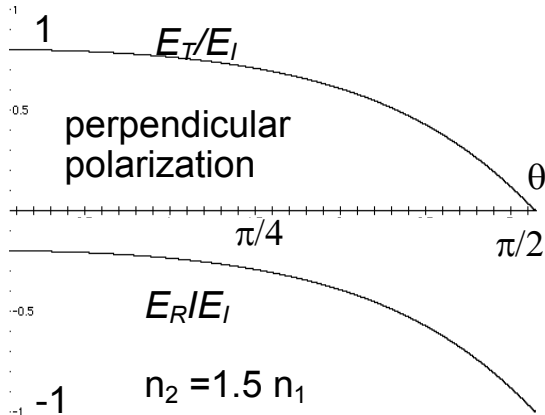
see "Fresnel Equations"

$$E_{T0\perp} = E_{I0\perp} \times \frac{2 n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$E_{T0\parallel} = E_{I0\parallel} \times \frac{2 \cos\theta_1/n_2}{\cos\theta_1/n_1 + \cos\theta_2/n_2}$$

$$E_{R0\perp} = E_{I0\perp} \times \frac{n_1 \cos\theta_1 - n_2 \cos\theta_2}{n_1 \cos\theta_1 + n_2 \cos\theta_2}$$

$$E_{R0\parallel} = E_{I0\parallel} \times \frac{\cos\theta_2/n_2 - \cos\theta_1/n_1}{\cos\theta_1/n_1 + \cos\theta_2/n_2}$$



Brewster angle:

$E_{R0\parallel}$ changes sign

$$\cos\theta_1/n_1 = \cos\theta_2/n_2$$

$$\Rightarrow \sin\theta_1 \cos\theta_1 = \sin\theta_2 \cos\theta_2$$

$$\Rightarrow \theta_1 + \theta_2 = \pi/2$$

$$\text{or } \sin \theta_1 = \sin \theta_B = \frac{n_2 / n_1}{\sqrt{1 + (n_2 / n_1)^2}}$$

$$\Rightarrow \tan \theta_B = \frac{n_2}{n_1}$$

5.3. Absorption and Dispersion

5.3.A. WAVES IN CONDUCTORS

linearly polarized plane wave \Rightarrow

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)},$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t - \theta)}$$

General Forms

Good Conductors

$\left. \begin{aligned} \nabla^2 \mathbf{E} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0 \\ \nabla^2 \mathbf{H} - \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} = 0 \end{aligned} \right\}$	wave equations diffusion equations \rightarrow	$\left\{ \begin{aligned} \nabla^2 \mathbf{E} - \sigma\mu \frac{\partial \mathbf{E}}{\partial t} = 0 \\ \nabla^2 \mathbf{H} - \sigma\mu \frac{\partial \mathbf{H}}{\partial t} = 0 \end{aligned} \right.$
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$\mathbf{k}\cdot\mathbf{E} = 0, \nabla\cdot\mathbf{B} \Rightarrow \mathbf{k}\cdot\mathbf{H} = 0$	transversality	$\mathbf{k}\cdot\mathbf{E} = 0, \mathbf{k}\cdot\mathbf{H} = 0$
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$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}$	$\mathbf{E} \perp \mathbf{H}$	$\mathbf{k} \times \mathbf{E} = \omega\mu\mathbf{H}$
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$k^2 = \epsilon\mu\omega^2 + i\sigma\mu\omega = (k e^{i\theta})^2$	$k(\omega)^2$	$k^2 = i\sigma\mu\omega, \theta = \pi/4$
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$k = k + i\kappa$	$k(\omega)$	$k = \sqrt{\frac{\omega\mu\sigma}{2}}(1+i)$
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$\lambda \equiv 2\pi/\text{Re}k = 2\pi / k$	wavelength	$\lambda = 2\pi \sqrt{2/\omega\mu\sigma}$
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$\delta \equiv 1/\text{Im}k = 1/\kappa$	attenuation length ("skin depth")	$\delta = \sqrt{2/\omega\mu\sigma}$
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$v_p \equiv \omega / k$	phase velocity	$v_p = \sqrt{2\omega/\mu\sigma}$
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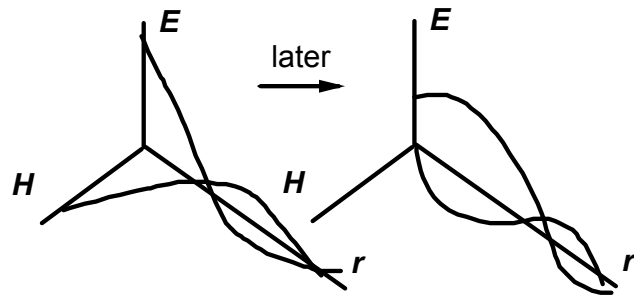
$n \equiv \frac{c}{v_p}$	index of refraction	$n = c\sqrt{\sigma\mu/2\omega}$
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Wave propagates in the z direction

$$\text{Re } \mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{-z/\delta} \cos(z/\delta - \omega t)$$

$$\text{Re } \mathbf{H}(\mathbf{r}, t) =$$

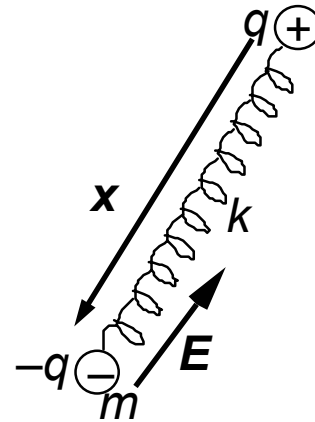
$$\left(\frac{\sigma}{\omega\mu}\right)^{\frac{1}{2}} \mathbf{E}_0 e^{-z/\delta} \cos\left(z/\delta - \omega t + \frac{\pi}{4}\right)$$



note $\sigma \gg \omega\mu \Rightarrow H \gg E$

5.3.B. FREQUENCY DEPENDENT PERMITTIVITY

Wave-matter interaction: time dependent polarization of damped oscillators

static charges $\pm q$ separation $\equiv \mathbf{x}$ dipole moment $= -q\mathbf{x} \equiv \mathbf{p}$ reduced mass m , spring constant k vibration frequency $\omega_0 = \sqrt{k/m}$ electric field $= \mathbf{E}(t) = \mathbf{E}_0 e^{i\omega t}$ forces on negative charge:

electric force restoring force damping force

$$\mathbf{F} = m\mathbf{a} = m \frac{\partial^2 \mathbf{x}}{\partial t^2} = -q\mathbf{E}(t) - m\omega_0^2 \mathbf{x} - m\gamma \frac{\partial \mathbf{x}}{\partial t},$$

solution $\mathbf{x}(t) = \mathbf{x}_0 e^{-i\omega t} \rightarrow m\omega^2 \mathbf{x}_0 = q\mathbf{E}_0 + m\omega_0^2 \mathbf{x}_0 + im\omega\gamma \mathbf{x}_0$

$$\rightarrow \mathbf{x}_0 = -\frac{q}{m} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \mathbf{E}_0 \Rightarrow \text{dipole moment } \mathbf{p} = -q\mathbf{x}$$

$$\rightarrow N \frac{\text{oscillators}}{\text{volume}} \Rightarrow \text{polarization density } \mathbf{P} = N\mathbf{p} = \chi(\omega)\epsilon_0 \mathbf{E}$$

$$\rightarrow \chi(\omega) = \text{electric susceptibility} = \frac{Nq^2}{m\epsilon_0} \frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega}$$

$$\rightarrow \text{In general, } \epsilon_R(\omega) = 1 + \chi(\omega) = 1 + \frac{Nq^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}$$

Plane wave propagating in the z direction, $\mathbf{E}(z, t) = \mathbf{E}_0 e^{i(kz - \omega t)}$ with the complex wave number $k(\omega) = k(\omega) + i\kappa(\omega) = \sqrt{\epsilon_R(\omega)} \frac{\omega}{c}$ (i) absorption coefficient, $\alpha(\omega) \equiv 2\kappa$ (ii) index of refraction, $n(\omega) = k \frac{c}{\omega}$

$$\alpha = \frac{Nq^2\omega^2}{m\epsilon_0 c} \sum_j \frac{f_j \gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \quad n = 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2}$$