Capstones in Physics: Electromagnetism

6. RADIATION

1. Potential Formulation

2. Oscillating electric dipole

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6.1. POTENTIAL FORMULATION

6.1.A. SCALAR AND VECTOR POTENTIALS

electrostatics: 3 components of $\boldsymbol{E}$ not independent, $\nabla\times\boldsymbol{E}=0$

only 1 function's worth of information, $V(\boldsymbol{r}) \rightarrow \boldsymbol{E} = -\nabla V(\boldsymbol{r})$

electrodynamics: 6 comp's of $\boldsymbol{E}$ and $\boldsymbol{B}$ not independent,

two homogeneous equations without sources

$\nabla\cdot\boldsymbol{B} = 0$ is a scalar eq.; $\nabla\times\boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$ has 3 components

$\Rightarrow$ 4 constraints on the 6 components of $\boldsymbol{E}$ and $\boldsymbol{B}$,

$\Rightarrow$ only 2 independent functions!

We look for a more efficient description, also with 4-vectors

How can we generalize the electrostatic potential $V(\boldsymbol{r})$?

We need an extra potential to obtain $\nabla\times\boldsymbol{E} = -\dfrac{\partial \boldsymbol{B}}{\partial t}$, since $\nabla\times(-\nabla V(\boldsymbol{r}))=0$

Because $\nabla \cdot \boldsymbol{B} = 0$,

> there exists a vector function $\boldsymbol{A}$ such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$,
>
> then $\boldsymbol{E} = -\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t}$ has the right curl,

and also the right divergence since $\nabla \cdot (\nabla\times\boldsymbol{A}) = 0$ for any $\boldsymbol{A}$

$\nabla \cdot\boldsymbol{E} = \rho/\varepsilon_0 \quad\Rightarrow\quad \nabla^2 V + \dfrac{\partial}{\partial t}(\nabla\cdot\boldsymbol{A}) = -\dfrac{1}{\varepsilon_0}\rho$

$\nabla\times\boldsymbol{B} - \dfrac{1}{c^2}\dfrac{\partial \boldsymbol{E}}{\partial t} = \mu_0\boldsymbol{J} \quad\Rightarrow\quad \left(\nabla^2\boldsymbol{A} - \dfrac{1}{c^2}\dfrac{\partial^2\boldsymbol{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\boldsymbol{A} + \dfrac{1}{c^2}\dfrac{\partial V}{\partial t}\right) = -\mu_0\boldsymbol{J}$

6.1.B. GAUGE TRANSFORMATIONS

Note: ***A*** is not unique: local gauge invariance

Let ***A'*** = ***A*** + ∇f, then $\nabla \times$ ***A'*** = $\nabla \times$ ***A*** $\Rightarrow$ same ***B***

Also let *V'* = *V* – $\dfrac{\partial f}{\partial t}$, $\Rightarrow$ same ***E***

For any scalar function f, the changes in *V* and ***A*** (gauge transformation) will not affect the physical quantities ***E*** and ***B***.

<u>Coulomb Gauge</u>: $\nabla \cdot \mathbf{A} = 0 \Rightarrow \nabla^2 f = 0$

$\nabla^2 V = -\dfrac{1}{\varepsilon_0}\rho \Rightarrow V(\mathbf{r},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \dfrac{\rho(\mathbf{r'},t)}{|\mathbf{r}-\mathbf{r'}|}d^3\mathbf{r'}$: determined by $\rho(\mathbf{r},t)$ right now

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J} + \frac{1}{c^2}\nabla\left(\frac{\partial V}{\partial t}\right)$$

The scalar potential is particularly simple to calculate, but the vector potential is particularly difficult to calculate.

<u>Lorentz Gauge</u>: $\boxed{\nabla \cdot \mathbf{A} = -\dfrac{1}{c^2}\dfrac{\partial V}{\partial t}}$

Maxwell's Eqs. $\Rightarrow \nabla^2 V - \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} = -\dfrac{1}{\varepsilon_0}\rho$, $\nabla^2 \mathbf{A} - \dfrac{1}{c^2}\dfrac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}$

- 4-vector formalism

A and *V* form a 4-vector $\Rightarrow$ **A**(**r**) = (***A***(***r***,*t*), *V*(***r***,*t*)/*c*), **r** = (***r***,*ct*)

d'Alembertian derivative $\Rightarrow \Box = \left(\nabla, -\dfrac{\partial}{c\partial t}\right)$

current **J** = (***J***, $c\rho$) conserved $\Rightarrow \Box \cdot \mathbf{J} = 0$

Lorentz Gauge $\Rightarrow \Box \cdot \mathbf{A} = 0$

Maxwell's Eqs. $\Rightarrow \Box^2 \mathbf{A} = -\mu_0 \mathbf{J}$

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6.2. OSCILLATING ELECTRIC DIPOLE

$\boldsymbol{p}(t) = \boldsymbol{p}_O \cos \omega t$, $\boldsymbol{p}_O = Q_O \boldsymbol{s}$

complex notation:

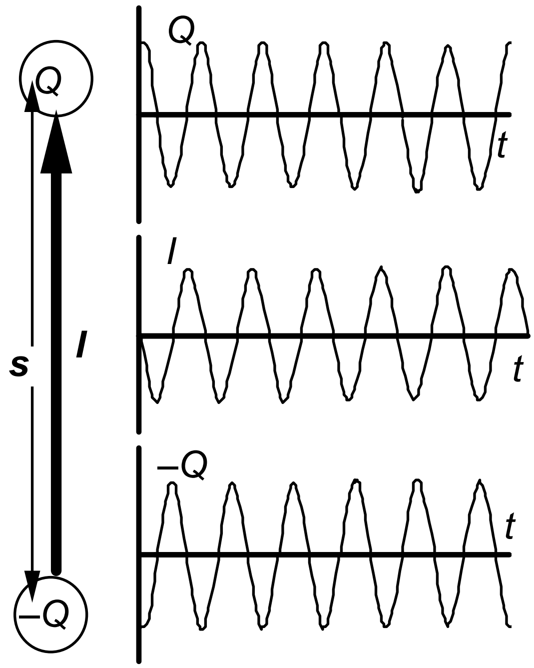
$$\boldsymbol{p}(t) = \boldsymbol{p}_O \exp i\omega t$$

current $I = \frac{dQ}{dt} = i\omega Q_O \exp i\omega t$

$$= I_O \exp i\omega t = \frac{ikc}{s}\, \boldsymbol{p}(t)$$

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{Q_i(T = t - |\mathbf{r} - \mathbf{R_i}|/c)}{|\mathbf{r} - \mathbf{R_i}|}$$

$$= \frac{Q_O}{4\pi\varepsilon_O} e^{i\omega t} \left\{ \frac{e^{-ik|\boldsymbol{r} - \boldsymbol{s}/2|}}{|\boldsymbol{r} - \boldsymbol{s}/2|} - (\boldsymbol{s} \to -\boldsymbol{s}) \right\}$$



$$\boldsymbol{A}(\mathbf{r}) = \frac{\mu_O}{4\pi} \int d^3\boldsymbol{R}\, \frac{\boldsymbol{J}(\boldsymbol{R}, T = t - |\boldsymbol{R} - \boldsymbol{r}|/c)}{|\boldsymbol{R} - \boldsymbol{r}|} = \frac{\mu_O}{4\pi} \int d\boldsymbol{L}\, \frac{I(\boldsymbol{L}, T = t - |\boldsymbol{r} - \boldsymbol{L}|/c)}{|\boldsymbol{r} - \boldsymbol{L}|}$$

$$= \frac{\mu_O I_O}{4\pi} \int_{-\boldsymbol{s}/2}^{\boldsymbol{s}/2} d\boldsymbol{L}\, \frac{e^{i\omega(t - |\boldsymbol{r} - \boldsymbol{L}|/c)}}{|\boldsymbol{r} - \boldsymbol{L}|} = \frac{\mu_O I_O}{4\pi} e^{i\omega t} \int_{-\boldsymbol{s}/2}^{\boldsymbol{s}/2} d\boldsymbol{L}\, \frac{e^{-ik|\boldsymbol{r} - \boldsymbol{L}|}}{|\boldsymbol{r} - \boldsymbol{L}|}$$

far away
<u>$s \ll r$</u> $\Rightarrow |\boldsymbol{r} - \boldsymbol{L}| = \sqrt{r^2 - 2rL\cos\theta + L^2} = r\sqrt{1 - 2\frac{L}{r}\cos\theta + \frac{L^2}{r^2}} \approx r - L\cos\theta$

$$V(\mathbf{r}) \approx \frac{Q(t - r/c)}{4\pi\varepsilon_O r} \left\{ \frac{e^{i\frac{ks}{2}\cos\theta}}{1 - \frac{s}{2r}\cos\theta} - (\boldsymbol{s} \to -\boldsymbol{s}) \right\},$$

$$\boldsymbol{A}(\mathbf{r}) \approx \frac{\mu_O \boldsymbol{I}(t - r/c)}{4\pi r} \int_{-s/2}^{s/2} dL\, \frac{e^{ikL\cos\theta}}{1 - \frac{L}{r}\cos\theta}$$

short dipole
$kL \ll 1$ $\Rightarrow V(\mathbf{r}) \approx \frac{\boldsymbol{r} \cdot \boldsymbol{p}(t - r/c)}{4\pi\varepsilon_O r^2} \left(\frac{1}{r} + ik \right)$ = static + radiation

$$\boldsymbol{A}(\mathrm{r}) \approx \frac{\mu_O \boldsymbol{I}(t - r/c)}{4\pi r}\, s = \frac{1}{c} \frac{\boldsymbol{p}(t - r/c)}{4\pi\varepsilon_O r}\, ik \quad \text{= pure radiation}$$

6.3.A. OSCILLATING ELECTRIC DIPOLE FIELDS

see “Oscillating Electric Dipole”

$\boldsymbol{p}(t) = \boldsymbol{p}_O \cos \omega t$, $\boldsymbol{p}_O = Q_O \boldsymbol{s}$

complex notation: $\boldsymbol{p}(t) = \boldsymbol{p}_O \exp i\omega t$

$$I = dQ/dt = i\omega Q_O \exp i\omega t = \frac{i\omega}{s}\, \boldsymbol{p}(t)$$

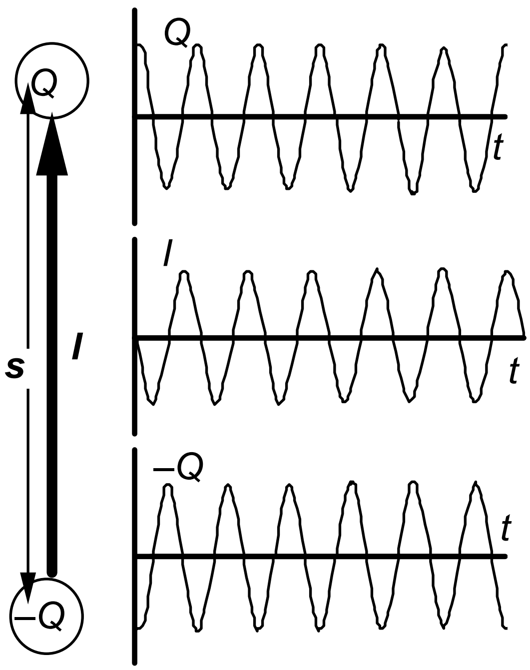
$s \ll \lambda \ll r \Rightarrow$

$$V(\mathbf{r},t) \approx \frac{\mathbf{r}\cdot\mathbf{p_0}}{4\pi\varepsilon_0 r^2} ike^{i\omega t}e^{-ikr}$$

$$= \frac{ikp_0 e^{i\omega t}}{4\pi\varepsilon_0}\frac{e^{-ikr}}{r}\cos\theta$$

$$\mathbf{A}(\mathbf{r},t) \approx \frac{\mathbf{p_0}}{4\pi c\varepsilon_0 r} ike^{i\omega t}e^{-ikr}$$

$$= \frac{ik\mathbf{p_0}e^{i\omega t}}{4\pi c\varepsilon_0}\frac{e^{-ikr}}{r}$$



Electric field: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$

$$\nabla V \propto \nabla\left(\frac{e^{-ikr}}{r}\cos\theta\right) = \hat{r}\left(-ik\frac{e^{-ikr}}{r} - \frac{e^{-ikr}}{r^2}\right)\cos\theta + \hat{\theta}\left(-\frac{e^{-ikr}}{r^2}\right)\sin\theta$$

$$\approx \hat{r}\left(-ik\frac{e^{-ikr}}{r}\cos\theta\right) \Rightarrow \nabla V = \hat{r}\left(\frac{k^2}{4\pi\varepsilon_0}\frac{p(t-r/c)}{r}\cos\theta\right)$$

$$\frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} = i\omega\mathbf{A}(\mathbf{r},t) = -\frac{k^2\mathbf{p}(t-r/c)}{4\pi\varepsilon_0 r}$$

$$\mathbf{E}(\mathbf{r},t) = -\frac{k^2}{4\pi\varepsilon_0 r}\frac{[\mathbf{p}(t-r/c)\cdot\mathbf{r}]\mathbf{r}}{r^2} + \frac{k^2\mathbf{p}(t-r/c)}{4\pi\varepsilon_0 r} = \frac{k^2}{4\pi\varepsilon_0 r}\left[\mathbf{p} - \frac{(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^2}\right]$$

$$= -\left[\frac{k^2 p(t-r/c)}{4\pi\varepsilon_0 r}\sin\theta\right]\hat{\theta}$$

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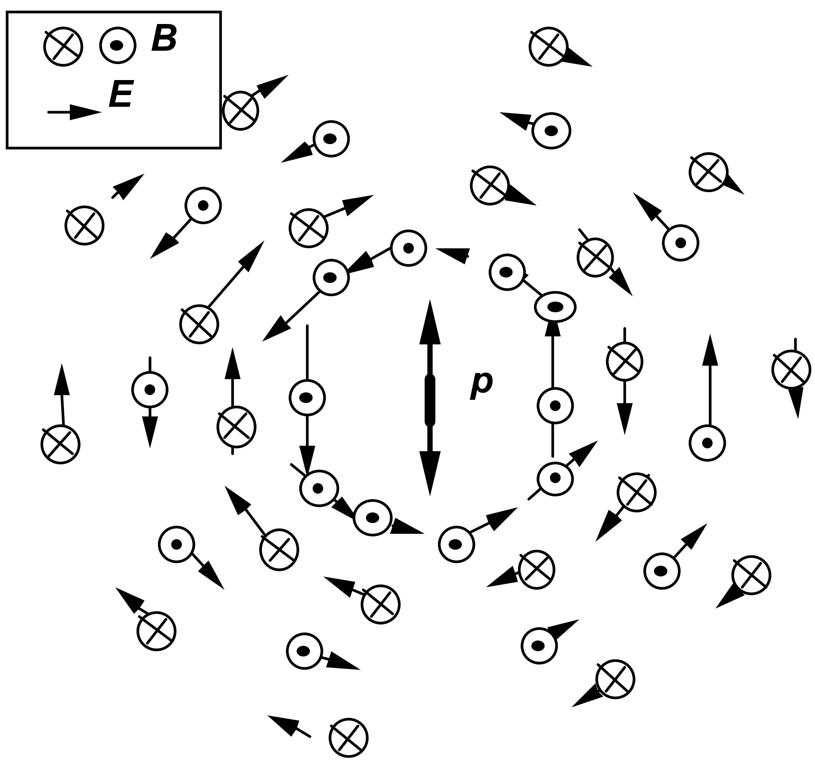
Magnetic field: $\mathbf{B} = \nabla \times \mathbf{A}$

$$\mathbf{B}(\mathbf{r},t) = \nabla \times \left(\frac{ik\mathbf{p_0}e^{i\omega t}}{\mathbf{4}\pi c\varepsilon_\mathbf{0}} \frac{e^{-ikr}}{r} \right) = -\frac{ik\mathbf{p_0}e^{i\omega t}}{\mathbf{4}\pi c\varepsilon_\mathbf{0}} \times \nabla \left(\frac{e^{-ikr}}{r} \right)$$

$$= \left(-\frac{ik\mathbf{p_0}e^{i\omega t}}{\mathbf{4}\pi c\varepsilon_\mathbf{0}} \right) \times \hat{r} \left(-ik\frac{e^{-ikr}}{r} - \frac{e^{-ikr}}{r^2} \right)$$

$$\approx \left(-\frac{ik\mathbf{p_0}e^{i\omega t}}{\mathbf{4}\pi c\varepsilon_\mathbf{0}} \right) \times \hat{r} \left(-ik\frac{e^{-ikr}}{r} \right)$$

$$\approx \frac{1}{c}\frac{k^2}{\mathbf{4}\pi\varepsilon_\mathbf{0} r}\frac{\mathbf{p}(t-r/c)\times\mathbf{r}}{r} = -\left[\frac{1}{c}\frac{k^2 p(t-r/c)}{\mathbf{4}\pi\varepsilon_\mathbf{0} r}\mathbf{\sin}\,\theta \right]\hat{\phi}$$



in phase with, perp. to ***B***, pattern moves outward with time

ELECTRIC DIPOLE POWER

see “Electric Dipole Radiation”

$$\boldsymbol{B}(\boldsymbol{r},t) \rightarrow \frac{1}{c} \frac{k^2}{4\pi\varepsilon_0 r} \frac{\boldsymbol{r} \times \boldsymbol{p_0}}{r} e^{i\omega(t-r/c)}$$

$$\boldsymbol{E}(\boldsymbol{r},t) \rightarrow \frac{k^2}{4\pi\varepsilon_o r} \left(\boldsymbol{p}_o - \frac{(\boldsymbol{r}\bullet\boldsymbol{p}_o)\boldsymbol{r}}{r^2}\right) e^{i\omega(t-r/c)}$$

$$\boldsymbol{S}_{av}(\boldsymbol{r}) = \frac{1}{2\mu_0} \boldsymbol{E}_o \times \boldsymbol{H}_o = \frac{k^4 p_o^2}{32\pi^2 c \varepsilon_o^2 \mu_o r^2} \hat{r} \sin^2\theta = \frac{\mu_o}{4\pi c} \frac{\omega^4 p_o^2}{8\pi r^2} \hat{r} \sin^2\theta$$

$\boldsymbol{S}_{av}$ is always radial outward, $\sim \frac{1}{r^2} \sin^2\theta$ radiation field

RADIATED POWER

$$P_{av} = \oint d\boldsymbol{a}\bullet\boldsymbol{S}_{av} = \frac{\mu_o}{4\pi c} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, r^2 \sin\theta \frac{\omega^4 p_o^2}{8\pi r^2} \sin^2\theta = \frac{\mu_o}{4\pi c} \frac{\omega^4 p_o^2}{3}$$

- proportional to $\omega^4 p_o^2$
- independent of r √

an evocative formula

use $p_o = I_o s/\omega$, $\omega = ck$ $\Rightarrow$ $P_{av} = \frac{\mu_o}{6\pi c} (ks)^2 I_{rms}^2 = R_{rad} I_{rms}^2$

SCATTERING OF LIGHT from small particles,

p_o ~ incident field, p_o^2 ~ intensity,

$$\frac{\text{scattered power}}{\text{incident intensity}} \sim \omega^4$$

$\Rightarrow$ blue light scattered more than red

$\Rightarrow$ blue sky, red sunset

