Capstones in Physics: Electromagnetism

4. ELECTRODYNAMICS

4.1. ELECTROMOTIVE FORCE

4.1.A CONDUCTIVITY AND RESISTANCE

linear isotropic conductor: current density $\mathbf{J} = \sigma \mathbf{E}$, σ = conductivity

(unit siemens/meter, 1 siemens = 1 ampere/volt)

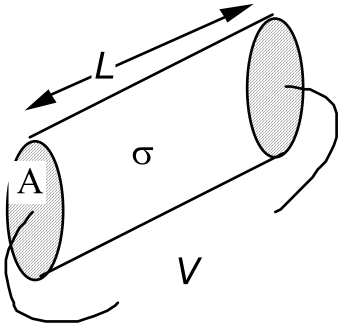
typical conductivity of metals ~ 10^7 siemens/m ,

solutions ~ 1 siemens/m

- in many nonmetallic solids, conduction is nonlinear (especially semiconductors)
- dielectric polarization → imaginary conductivity

idealization: an **electrode** is a piece of material with $\sigma \approx \infty$ fastened to another material

example:



a uniform cylinder with electrodes on ends

cross section A (shape irrelevant), length L ,

assume an applied potential V , then

$E = V/L$, $J = \sigma E$, $I = JA = V\sigma A/L$,

ratio $R = V/I = L/\sigma A$ depends only on geometry & materials

The **resistance** R between two electrodes is the ratio of

$$\frac{\text{their potential difference} V}{\text{the current } I \text{ flowing between them}} = \frac{V}{I} = R$$

(note analogy to capacitance, $C = \frac{Q}{V}$)

For linear materials R is independent of the applied voltage.

CYLINDRICAL RESISTOR

see "Resistance"

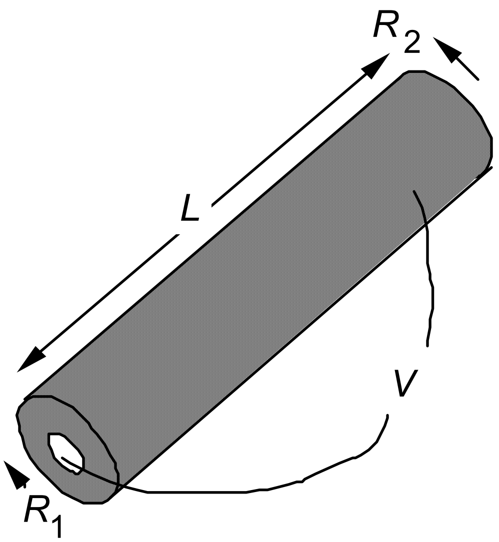
hollow tube

inner radius R_1,

outer radius R_2,

length L,

electrodes inside and outside



assume charge $Q = \lambda L$ on inner electrode $\Rightarrow$

radial electric field, magnitude $E(r) = \dfrac{\lambda}{2\pi\varepsilon r} = \dfrac{Q}{2\pi\varepsilon r\, L}$

current: $J(r) = \sigma E(r) = \dfrac{\sigma Q}{2\pi\varepsilon r\, L} \Rightarrow I = \oint dA \bullet \mathbf{J} = \dfrac{\sigma Q}{\varepsilon}$

potential difference: $V = -\int_{R_1}^{R_2} dr \;\; E(r) = -\dfrac{Q}{2\pi\varepsilon L} \ln (R_2/R_1)$

resistance: $R = \dfrac{V}{I} = \dfrac{1}{2\pi\sigma L} \ln(R_2/R_1)$

SPHERICAL RESISTOR

see "Resistance"

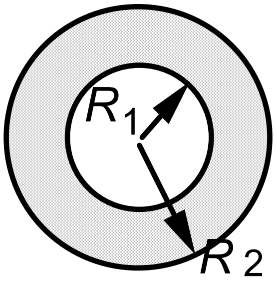
Concentric spherical shells,

inner radius = R_1,

outer radius = R_2 ,

conductivity of medium

between shells = σ



Assume a free charge Q on the inner shell,

then the potential in the region between the plates is

$$V(\mathbf{r}) = \frac{Q}{4\pi\varepsilon|\mathbf{r}|} ,$$

the potential difference between the plates is

$$\Delta V = V(R_1) - V(R_2) = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

the electric field in the region between the plates is

$$\mathbf{E}(\mathbf{r}) = -\nabla V = \frac{Q\hat{r}}{4\pi\varepsilon r^2}$$

the current density in the region between the plates is

$$\mathbf{J}(\mathbf{r}) = \sigma\mathbf{E}(\mathbf{r}) = \frac{\sigma Q\hat{r}}{4\pi\varepsilon r^2}$$

the total current is the integral of $\mathbf{J}(\mathbf{r})$ over a surface between the plates,

$$I = \oint dA\cdot\mathbf{J} = \sigma \oint dA\cdot\mathbf{E} = \sigma\frac{Q}{\varepsilon} \quad \text{using Gauss' law}$$

$$R = \frac{V}{I} = \frac{1}{4\pi\sigma}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

4.1.B DISSIPATED POWER

For an average electron, the power from the field is

$$P_e = \mathbf{F}\bullet\mathbf{v}_{\text{drift}} = (-e\mathbf{E})\bullet(\mathrm{M}\mathbf{E})$$

The power per unit volume is

$$n_c\mathrm{P}_e = n_c(-e)\mathrm{M}\,\mathbf{E}^2 = \sigma\mathbf{E}^2 = \mathbf{J}\bullet\mathbf{E}$$

What happens to this power?

It is **thermalized,** i.e. converted into heat!

The energy is shared by the electrons

- with each other and
- with the ions in the lattice

$$\underline{\text{Total dissipated power}} = \frac{\text{change in potential energy}}{\text{time}}$$

$$= \frac{\text{charge}}{\text{time}} \times \text{change in potential}$$

$$= I\ V$$

total power dissipated in a resistor: $P = IV = I^2R = V^2/R$

→ Joule heating law

check:

for rectangular resistor, $I = J \times$ area, $V = E \times$ thickness

$$P = I\ V = (J\ E) \times (\text{area} \times \text{thickness})$$

$$= (\text{power per unit volume}) \times (\text{volume}) \qquad \surd$$

4.1.C. ELECTROMOTIVE FORCE (emf)

Lorentz force on charge Q : $\boldsymbol{F}(\boldsymbol{r}) = Q\,[\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r})\,]$

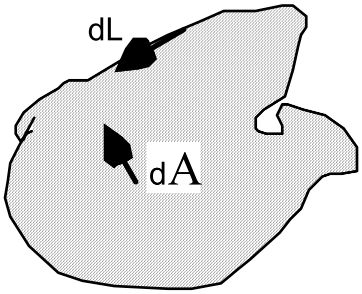
Work done by fields when particle moves $\mathrm{d}\boldsymbol{r} = \boldsymbol{v}\,\mathrm{d}t$:

$$\mathrm{d}W = \mathrm{d}\boldsymbol{r} \bullet \boldsymbol{F}(\boldsymbol{r}) = Q\,\mathrm{d}t\;[\boldsymbol{E}(\boldsymbol{r})\bullet\boldsymbol{v} + \boldsymbol{v}\bullet(\boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{r}))] = Q\,\mathrm{d}t\;\boldsymbol{E}(\boldsymbol{r})\bullet\boldsymbol{v} = Q\,\boldsymbol{E}(\boldsymbol{r})\bullet\mathrm{d}\boldsymbol{r}$$

because $\boldsymbol{v} \perp \boldsymbol{v} \times \boldsymbol{B}$ Power delivered to Q is $P = \dfrac{\mathrm{d}W}{\mathrm{d}t} = Q\,\boldsymbol{v} \bullet \boldsymbol{E}$

<u>Work Done on a Current Moving in a Circuit</u>

a *circuit* is a wire which starts and ends at the same place.



Power in a section dL of the circuit is

$$\begin{aligned}\mathrm{d}P &= \mathrm{d}Q\;\boldsymbol{v}\bullet\boldsymbol{E}(\boldsymbol{R})\\ &= (\rho\,\mathrm{d}S\,\mathrm{d}L)\;\boldsymbol{v}\bullet\boldsymbol{E}(\boldsymbol{R})\\ &= \mathrm{d}S\,\mathrm{d}L\;\boldsymbol{J}\bullet\boldsymbol{E}(\boldsymbol{R})\\ &= I\;\mathrm{d}\boldsymbol{L}\bullet\boldsymbol{E}(\boldsymbol{R})\end{aligned}$$

Total power from fields into the circuit $= P = \oint \mathrm{d}P = I \oint \mathrm{d}\boldsymbol{L}\bullet\boldsymbol{E}(\boldsymbol{R}) = I\,V_{\mathrm{emf}}$

where the *electromotance* (emf) is $V_{\mathrm{emf}} \equiv \oint \mathrm{d}\boldsymbol{L}\bullet\boldsymbol{E}(\boldsymbol{R})$

use Stokes Law: $V_{\mathrm{emf}} = \int \mathrm{d}\boldsymbol{S}\bullet(\nabla\times\boldsymbol{E}) = -\int \mathrm{d}\boldsymbol{S}\bullet\dfrac{\partial\boldsymbol{B}}{\partial t} \equiv -\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}$

Faraday: $V_{\mathrm{emf}} = -\dfrac{\mathrm{d}\Phi}{\mathrm{d}t}$ where $\Phi = \int \mathrm{d}\boldsymbol{S}\bullet\boldsymbol{B}$ $\equiv$ Magnetic Flux

Lenz's law: Induced currents counter change in flux

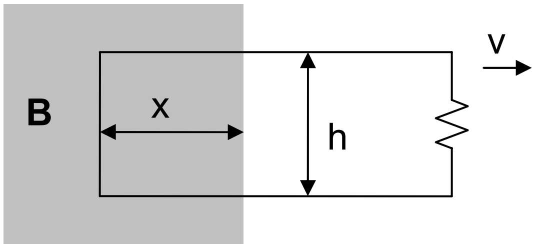
Conclusion: Faraday law works when circuit is moving

Example: Motional emf → generator

The magetic flux $\Phi = \int \boldsymbol{B}\cdot d\boldsymbol{A} = Bhx$

As the loop moves the fulx decreases:

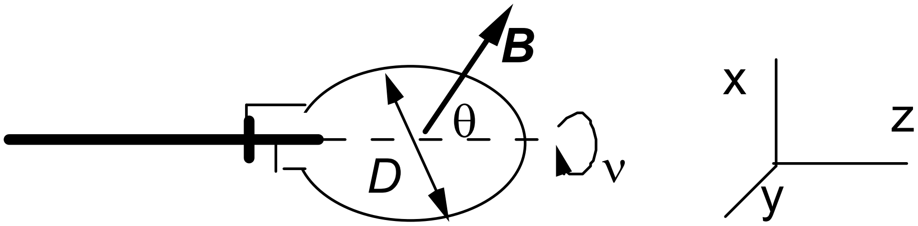
$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv \quad\Rightarrow\quad V_{emf} = Bhv$$



Example: magnetometer

An old-fashioned magnetometer consists of
a small circular coil of diameter D
at the end of a non-metallic probe.

The coil has N turns, and rotates about a diameter
with a frequency of ν revolutions per second.



(a) Find the induced voltage V(t) when the coil is in a field ***B*** where the angle between ***B*** and the axis of rotation is θ.

Choose z-axis as axis of rotation, ***B*** in x-z plane, then

$$\mathbf{B} = B\,[\hat{x}\ \sin\theta + \hat{z}\ \cos\theta],$$

$$\mathbf{S} = \frac{\pi}{4}\ D^2\,[\hat{x}\ \cos(\omega t+\phi) + \hat{y}\ \sin(\omega t+\phi)] \text{ where } \omega = 2\pi\nu$$

$$\Phi = N\,\mathbf{S}\bullet\mathbf{B} = \frac{\pi D^2}{4}\ NB\sin\theta\cos(\omega t+\phi)$$

$$V_{\text{emf}} = -\ \frac{d\Phi}{dt} = \frac{1}{2}\ \pi^2 D^2\ \nu NB\sin\theta\sin(\omega t+\phi)$$

(b) Find the maximum induced voltage for

ν = 1000 Hz, D = 1 cm, N = 100 turns,

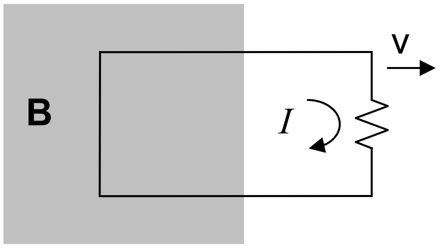
and B = 0.5 Gauss = 0.5×10^{-4} tesla,

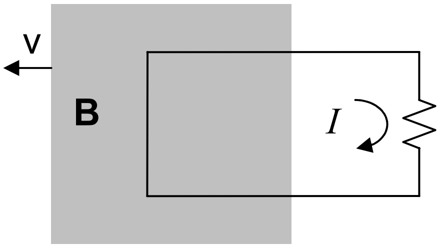
(approximately the magnetic field at the earth's surface)

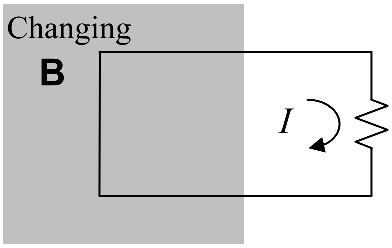
maximum $V_{\text{emf}} = \frac{\pi^2}{2}\ D^2\ \nu NB = 0.003V$

4.2 ELECTROMAGNETIC INDUCTION

4.2.A. FARADAY’S LAW







⇒ A changing magnetic field induces an electric field.

$$V_{\mathrm{emf}} \equiv \oint d\boldsymbol{L}\bullet\boldsymbol{E}(\boldsymbol{R}) = -\frac{d\Phi}{dt}$$

Stokes Law: $\int d\boldsymbol{S}\bullet(\nabla\times\boldsymbol{E}) = -\int d\boldsymbol{S}\bullet\frac{\partial\boldsymbol{B}}{\partial t} \Rightarrow \nabla\times\vec{E} = \frac{\partial\vec{B}}{\partial t}$

4.2.B. INDUCTANCE

MUTUAL INDUCTANCE

Magnetic field of one circuit or circuit element *a* may induce an emf in another element *b*.

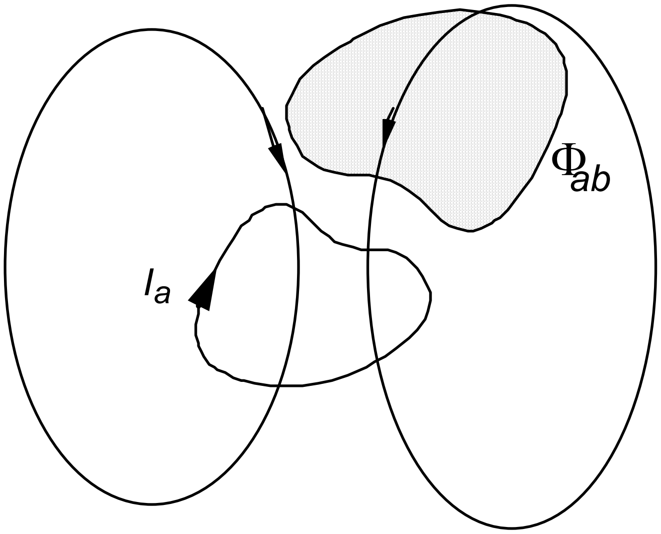
flux linking *b* due to *a* =

$\Phi_{ab} = M_{ab} I_a$

defines a geometric quantity $M = M_{ab} = M_{ba}$

To find *M*, compute flux:

$$\Phi_{ab} = \int d\mathbf{S}_b \cdot \mathbf{B}_a = \int d\mathbf{S}_b \cdot (\nabla \times \mathbf{A}_a) = \oint d\mathbf{L}_b \cdot \mathbf{A}_a(\mathbf{r}_b)$$



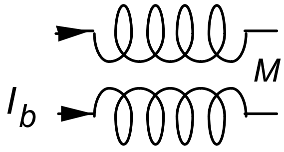
$$\text{voltage induced in } b = V_b = -\frac{d\Phi}{dt} = -M\frac{dI_a}{dt} - I_a\frac{dM}{dt},$$

$$\text{usually } \frac{dM}{dt} = 0 \Rightarrow \quad V_b = -M\frac{dI_a}{dt}, \quad \text{similarly } V_a = -M\frac{dI_b}{dt}$$

<u>complex notation:</u> $V_b = -i\omega M I_a$, $V_a = -i\omega M I_b$

Circuit notation: define currents I_a, I_b

M > 0 if currents give same sign of flux



<u>Example</u>

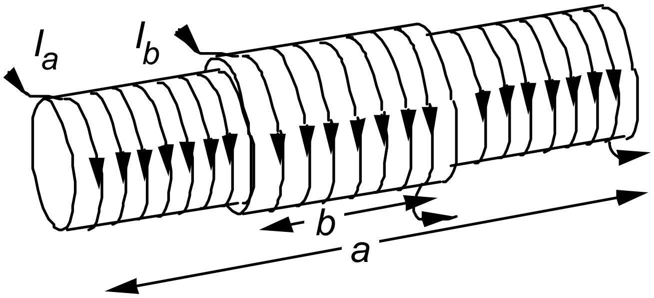
COAXIAL SOLENOIDS, radius *R*

find flux through *b* due to *a* :

Ampére's law: $B_a = \mu_o N_a I_a / a$,

Integrate $\Phi = \int \mathbf{B} \cdot d\mathbf{S}$

$= \pi R^2 N_b B_a = M_{ab} I_a$



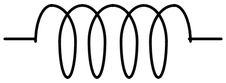
substitute $\Rightarrow M = \mu_o N_a N_b \pi R^2 / a$

SELF-INDUCTANCE

see “Mutual Inductance”

current in circuit or circuit element makes a magnetic field,

which may ⇒ induced emf in the circuit or circuit element itself.



self-inductance L

geometric quantity, definition:

flux linking circuit due to its own current = $\Phi = LI$.

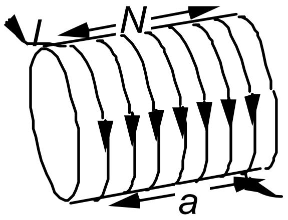
$$\text{Induced voltage } V = -\frac{d\Phi}{dt} = -L\,\frac{dI}{dt} - I\,\frac{dL}{dt}\ ;$$

$$\text{usually } \frac{dL}{dt} = 0 \Rightarrow V = -L\,\frac{dI}{dt}$$

sign from Lenz’s law: induced voltage encourages a current which would oppose the change in the flux.

complex notation $V = i\omega L\, I$, $Z = i\omega L$

self-inductance of a long solenoid

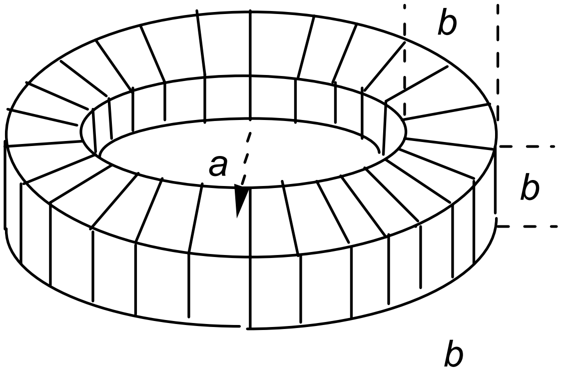


radius R , length a

$B = \mu_0 NI/a$, $\Phi = \pi R^2 N B = L\,I$

$$\Rightarrow L = \mu_0 N^2\, \pi R^2/a$$

toroid of square cross section



The current causes

$$H = \frac{N\,I}{2\pi r}\ ,\ B = \mu_0 H$$

$\Phi =$

$$N \int_0^b dz \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} dr\, B(r) = \frac{\mu_0 N^2 I b}{2\pi} \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} \frac{dr}{r}\ ,\quad L = \Phi/I = \frac{\mu_0 N^2 b}{2\pi}\ \ln\left(\frac{a+\frac{b}{2}}{a-\frac{b}{2}}\right)$$

4.2.C. INDUCTIVE ENERGY AND FORCE

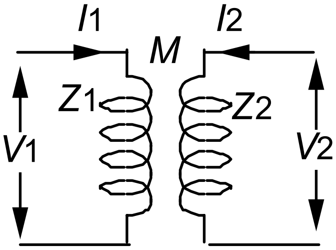
To build a current I in a fixed inductor,

a voltage $V = \frac{d\Phi}{dt} = L\frac{dI}{dt}$ must be applied to oppose the induced emf.

This stores energy in the inductor at a rate

$$\frac{dE}{dt} = P = IV = IL\frac{dI}{dt} = \frac{d}{dt}\left(\tfrac{1}{2}L I^2\right) \qquad \text{conclude} \quad E = \tfrac{1}{2} L I^2$$

Case of an inductor
with two windings:



power supplied by circuit 1 = $I_1V_1 = I_1[L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt}]$

power supplied by circuit 2 = $I_2V_2 = I_2[L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt}]$

$$\frac{dE}{dt} = \text{power supplied by circuits} = I_1[L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt}] + I_2[L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt}]$$

$$= \frac{d}{dt}\left(\frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2\right)$$

$$\boxed{\Rightarrow E_{\text{stored}} = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + MI_1I_2}$$

If two flux-linked circuits move with respect to each other,

force on relative coordinate $\boldsymbol{r}_{12}$: $F_x = I_1I_2\frac{\partial M}{\partial x}$, $F_y = I_1I_2\frac{\partial M}{\partial y}$,...

the torque on each circuit about an axis is $\tau = I_1I_2\frac{\partial M}{\partial \theta}$

where θ is the angle of rotation about the axis

4.3. MAXWELL EQUATIONS FOR *E* AND *B*

The electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ are defined as the quantitities which appear in the Lorentz force law for the force on a charge q moving with velocity $\boldsymbol{v}$:

$$\boldsymbol{F}(\boldsymbol{r}, \boldsymbol{v},t) = q \{\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r},t)\}$$

For time-independent fields we had four equations for $\boldsymbol{E}$ and $\boldsymbol{B}$:

differential equation		integral equation
$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_o$	Gauss' Law	$\oint \boldsymbol{E} \cdot d\boldsymbol{S} = q_{enclosed}/\varepsilon_o$
$\nabla \cdot \boldsymbol{B} = 0$		$\oint \boldsymbol{B} \cdot d\boldsymbol{S} = 0$
$\nabla \times \boldsymbol{E} = 0$	conservative force	$\oint \boldsymbol{E} \cdot d\boldsymbol{L} = 0$
$\nabla \times \boldsymbol{B} = \mu_o \boldsymbol{J}$	Ampere's Law	$\oint \boldsymbol{B} \cdot d\boldsymbol{L} = \mu_o I_{enclosed}$

When the fields are time dependent:

- the scalar equations $\nabla\cdot$ are unchanged.
- the vector equations $\nabla\times$ get additional terms

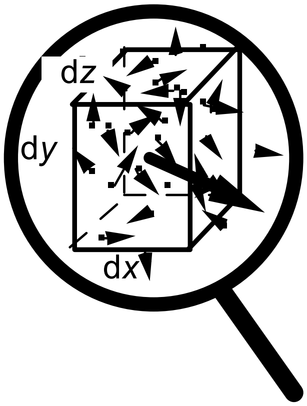
The general equations are

differential	MAXWELL'S EQUATIONS	integral
$\nabla \cdot \boldsymbol{E} = \rho/\varepsilon_o$	Gauss	$\oint d\boldsymbol{S} \cdot \boldsymbol{E} = Q_{enclosed}/\varepsilon_o$
$\nabla \cdot \boldsymbol{B} = 0$		$\oint d\boldsymbol{S} \cdot \boldsymbol{B} = 0$
$\nabla \times \boldsymbol{E} + \dfrac{\partial \boldsymbol{B}}{\partial t} = 0$	Faraday	$\oint d\boldsymbol{L} \cdot \boldsymbol{E} = -\dfrac{d}{dt} \int d\boldsymbol{S} \cdot \boldsymbol{B} \equiv -\dfrac{d\Phi}{dt}$
$\nabla \times \boldsymbol{B} - \dfrac{1}{c^2}\dfrac{\partial \boldsymbol{E}}{\partial t} = \mu_o \boldsymbol{J}$	Ampere	$\oint d\boldsymbol{L} \cdot \boldsymbol{B} = \mu_o \int dA \cdot \left(\boldsymbol{J} + \varepsilon_o \dfrac{\partial \boldsymbol{E}}{\partial t}\right)$

where $c \equiv (\varepsilon_o \mu_o)^{-1/2}$

4.4. SOURCES IN MAXWELL'S EQUATIONS

4.4.A. CONSERVATION OF CHARGE



current density$\boldsymbol{J}_{\text{charge}}(\boldsymbol{r})$

flow of charge through surface element dA is

$$dI = \boldsymbol{J}_{\text{charge}} \bullet d\boldsymbol{S} \quad \text{where } d\boldsymbol{S} \text{ is } \perp \text{ to surface}$$

$$\boldsymbol{J}_{\text{charge}} = \rho_{\text{charge}}\, \boldsymbol{v}_{\text{charge}}$$

$$= q_{\text{av}} n_{\text{charge}}\, \boldsymbol{v}_{\text{charge}}$$

where $\begin{pmatrix} n \\ v \end{pmatrix}$ = average $\begin{pmatrix} \text{density} \\ \text{velocity} \end{pmatrix}$ of charges

average weighted by amount of charge (including sign!)

consider the charge in an arbitrary volume: charge inside = $\int d\mathit{vol}\ \rho(\boldsymbol{r})$

charge leaving per second = $\oint d\boldsymbol{S}\bullet\boldsymbol{J}_{\text{charge}}(\boldsymbol{r})$

divergence theorem: $\int_S d\mathit{vol}\, \nabla\bullet\boldsymbol{V}(\boldsymbol{r}) = \oint_s d\boldsymbol{S}\bullet\boldsymbol{V}(\boldsymbol{r})$

apply with $\boldsymbol{V} \to \boldsymbol{J}_{\text{charge}}$: charge leaving per second $= \int_S d\mathit{vol}\, \nabla\bullet\boldsymbol{J}_{\text{charge}}(\boldsymbol{r})$

conservation of charge ⇒

rate of charge leaving = –rate of change of charge inside

$$\int_S d\mathit{vol}\, \nabla\bullet\boldsymbol{J}_{\text{charge}}(\boldsymbol{r}) = -\frac{d}{dt}\int d\mathit{vol}\ \rho(\boldsymbol{r}) = -\int d\mathit{vol}\ \frac{\partial\rho(\boldsymbol{r})}{\partial t}$$

integrals equal for arbitrary volume ⇒ integrands equal

$$\boxed{\nabla\bullet\boldsymbol{J}_{\text{charge}}(\boldsymbol{r}) = -\frac{\partial\rho(\boldsymbol{r})}{\partial t} \ \text{: Continuity equation}}$$

check Maxwell eqs

$$\mu_o\nabla\bullet\boldsymbol{J}_{\text{ch}} = \nabla\bullet\left(\nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\bullet\boldsymbol{E}) = -\frac{1}{c^2}\frac{\partial}{\partial t}\frac{\rho}{\varepsilon_0}\surd$$

Ampere div curl = 0 Gauss

4.4.B. DISPLACEMENT CURRENT

$$\nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_o \boldsymbol{J} \qquad \text{Ampere} \oint d\boldsymbol{L} \cdot \boldsymbol{B} = \mu_o \int d\boldsymbol{S} \cdot (\boldsymbol{J} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t})$$

where $c \equiv (\varepsilon_o \mu_o)^{-1/2}$

When the electric field is time dependent there is an

extra source of magnetic field: $\boldsymbol{J} \rightarrow \boldsymbol{J} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$

this source is called the **displacement current:** $\boldsymbol{J}_{\text{displacement}} \equiv \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t}$

Three parts of current density (*J*) in matter:

1. Motion of free charges $\boldsymbol{J}_f$
2. Bound current $\boldsymbol{J}_b = \nabla \times \boldsymbol{M}$
3. Polarization current $\boldsymbol{J}_p = \frac{\partial \boldsymbol{P}}{\partial t}$

The total charge density separated into two parts:

1. Free charges ρ_f
2. Bound charges $\rho_b = -\nabla \cdot \boldsymbol{P}$

4.4.C. ELECTRODYNAMICS IN MEDIA

differential	MAXWELL'S EQUATIONS	integral
$\nabla\cdot\boldsymbol{B} = 0$		$\oint d\boldsymbol{S}\cdot\boldsymbol{B} = 0$
$\nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0$		$\oint d\boldsymbol{L}\cdot\boldsymbol{E} = -\frac{d}{dt}\int d\boldsymbol{S}\cdot\boldsymbol{B} \equiv -\frac{d\Phi}{dt}$
$\nabla\cdot\boldsymbol{E} = \rho/\varepsilon_o$		$\oint d\boldsymbol{S}\cdot\boldsymbol{E} = Q_{\text{enclosed}}/\varepsilon_o$
$\nabla\times\boldsymbol{B} - \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} = \mu_o\boldsymbol{J}$		$\oint d\boldsymbol{L}\cdot\boldsymbol{B} = \mu_o\int dA\cdot(\boldsymbol{J} + \varepsilon_o\frac{\partial\boldsymbol{E}}{\partial t})$
$\nabla\cdot\boldsymbol{J} + \partial\rho/\partial t = 0$		$\oint dA\cdot\boldsymbol{J} = -\partial Q_{\text{enclosed}}/\partial t$

MACROSCOPIC FIELDS IN MATTER

• Polarization density of electric dipole moments	$\boldsymbol{P}$
• Polarization density of magnetic dipole moments	$\boldsymbol{M}$
• Displacement field	$\boldsymbol{D} \equiv \varepsilon_o\boldsymbol{E} + \boldsymbol{P}$
• Polarization charge	$\rho_{\text{bound}} \equiv -\nabla\cdot\boldsymbol{P}$
• Free charge	$\rho_{\text{free}} \equiv \rho - \rho_{\text{bound}}$
• Polarization current	$\boldsymbol{J}_{\text{pol}} \equiv \partial\boldsymbol{P}/\partial t$
• Current	$\boldsymbol{J}_{\text{free}} \equiv \boldsymbol{J} - \boldsymbol{J}_{\text{pol}} - \nabla\times\boldsymbol{M}$
• Field strength	$\boldsymbol{H} \equiv \boldsymbol{B}/\mu_o - \boldsymbol{M}$

differential	MAXWELL'S EQUATIONS IN MATTER	integral
$\nabla\cdot\boldsymbol{B} = 0$		$\oint d\boldsymbol{S}\cdot\boldsymbol{B} = 0$
$\nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0$	(Faraday)	$\oint d\boldsymbol{L}\cdot\boldsymbol{E} = -\frac{d}{dt}\int d\boldsymbol{S}\cdot\boldsymbol{B}$
$\nabla\cdot\boldsymbol{D} = \rho_{\text{free}}$	(Gauss)	$\oint d\boldsymbol{S}\cdot\boldsymbol{D} = Q_{\text{free}}$
$\nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t} = \boldsymbol{J}_{\text{free}}$	(Ampère)	$\oint d\boldsymbol{L}\cdot\boldsymbol{H} = \int d\boldsymbol{S}\cdot(\boldsymbol{J}_{\text{free}} + \frac{\partial\boldsymbol{D}}{\partial t})$
$\nabla\cdot\boldsymbol{J}_{\text{free}} + \partial\rho_{\text{free}}/\partial t = 0$	(charge cons)	$\oint d\boldsymbol{S}\cdot\boldsymbol{J}_{\text{free}} = -\partial Q_{\text{free}}/\partial t$

Linear media: $\boldsymbol{P} = \varepsilon_o\chi_e\boldsymbol{E}$, $\boldsymbol{D} = \varepsilon\boldsymbol{E} = (1+\chi_e)\varepsilon_o\boldsymbol{E}$

(but ferromagnets aren't linear) $\mathbf{M} = \chi_m\mathbf{H}$, $\mathbf{B} = \mu\mathbf{H} = (1+\chi_m)\mu_o\mathbf{H}$

4.5. FIELD ENERGY AND FORCES

Recall the potential energy of a static charge distribution,

$E_{\text{potential}} = \int d^3\boldsymbol{r}\ \frac{1}{2}\ \boldsymbol{D}(\boldsymbol{r})\cdot\boldsymbol{E}(\boldsymbol{r})$see "Field Energy"

The argument involved assembling the **charges**

- by bringing them in from infinity, or
- by gradually increasing the charge distribution everywhere

In each case there would be **currents**.

(conservation of charge ⇒ $\nabla\cdot\boldsymbol{J} = -\partial\rho/\partial t$)

These currents arise from the motion of charges,

and have their own kinetic energy.

If the charges are still moving at the end of the process,

there will be a **kinetic energy** due to their motion
in addition to the **potential energy** in the electric fields
and an **energy current** due to the transport of energy

linear case: kinetic energy density = $\rho_{KE}(\boldsymbol{r}) = \frac{1}{2}\ \boldsymbol{B}(\boldsymbol{R})\cdot\boldsymbol{H}(\boldsymbol{R})$

general case: energy current density = $\boldsymbol{S}(\boldsymbol{R}) = \boldsymbol{E}(\boldsymbol{R})\times\boldsymbol{H}(\boldsymbol{R})$
d(kinetic energy density) = $d\rho_{KE}(\boldsymbol{r}) = \boldsymbol{H}(\boldsymbol{r})\cdot d\boldsymbol{B}(\boldsymbol{r})$
d(potential energy density) = $d\rho_{PE}(\boldsymbol{r}) = \boldsymbol{E}(\boldsymbol{r})\cdot d\boldsymbol{D}(\boldsymbol{r})$

4.5.A. ENERGY CONSERVATION WITH FIELDS

Lorentz force on a charge Q is $\mathbf{F}(\mathbf{r}) = Q\,[\mathbf{E}(\mathbf{r}) + \mathbf{v}\times\mathbf{B}(\mathbf{r})]$

Work done by the fields

when particle moves by $d\mathbf{r} = \mathbf{v}\,dt$ is

$$dW = d\mathbf{r}\cdot\mathbf{F}(\mathbf{r}) = Q\,dt\,[\mathbf{E}(\mathbf{r})\cdot\mathbf{v} + \mathbf{v}\cdot(\mathbf{v}\times\mathbf{B}(\mathbf{r}))] \qquad = Q\,dt\;\mathbf{E}(\mathbf{r})\cdot\mathbf{v}$$

(vanishes, since $\mathbf{v}\perp\mathbf{v}\times\mathbf{B}$) †

Power exerted on Q : $P = dW/dt = Q\,\mathbf{v}\cdot\mathbf{E}$

Power delivered by the fields to a moving charge dQ_{free} :

$$dP = dQ_{\text{free}}\,\mathbf{v}\cdot\mathbf{E}(\mathbf{r}) = \rho_{\text{free}}\,d\mathit{vol}\,\mathbf{v}\cdot\mathbf{E}(\mathbf{r}) = d^3\mathbf{r}\;\mathbf{J}_{\text{free}}\cdot\mathbf{E}(\mathbf{r})$$

Rate of energy lost by the fields: $-dU/dt = \int d^3\mathbf{r}\;\mathbf{J}_{\text{free}}\cdot\mathbf{E}(\mathbf{r})$

Ampere's Law: $= \int d^3\mathbf{r}\,\left[\nabla\times\mathbf{H}(\mathbf{r}) - \frac{\partial\mathbf{D}}{\partial t}\right]\cdot\mathbf{E}(\mathbf{r})$

give names to terms: $= -\mathbf{P}_{\text{magnetic}} - \mathbf{P}_{\text{electric}}$

vector identity: $\mathbf{P}_{\text{m}} = -\int d^3\mathbf{r}\;\mathbf{H}(\mathbf{r})\cdot[\nabla\times\mathbf{E}(\mathbf{r})] + \int d^3\mathbf{r}\;\nabla\cdot[\mathbf{E}(\mathbf{r})\times\mathbf{H}(\mathbf{r})]$

Faraday: $\mathbf{P}_{\text{m}} = \int d^3\mathbf{r}\;\mathbf{H}(\mathbf{r})\cdot\left[\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r})\right] + \int d^3\mathbf{r}\;\nabla\cdot[\mathbf{E}(\mathbf{r})\times\mathbf{H}(\mathbf{r})]$

divergence thm: $= \int d^3\mathbf{r}\,\mathbf{H}(\mathbf{r})\cdot\frac{\partial\mathbf{B(r)}}{\partial t} + \oint d^2\mathbf{r}\,[\mathbf{E}(\mathbf{r})\times\mathbf{H}(\mathbf{r})]$

notation: $$\frac{dU}{dt} = \frac{dU_E}{dt} + \frac{dU_M}{dt} + \int \mathbf{S}\cdot dA$$

where the energy current density is $\mathbf{S}(\mathbf{r}) = \mathbf{E}(\mathbf{r})\times\mathbf{H}(\mathbf{r})$

and $dE_{\text{kin}} = \int d^3\mathbf{r}\;\mathbf{H}(\mathbf{r})\cdot d\mathbf{B}(\mathbf{r})$ $\quad dE_{\text{pot}} = \int d^3\mathbf{r}\;\mathbf{E}(\mathbf{r})\cdot d\mathbf{D}(\mathbf{r})$

d(kinetic energy density) = $d\rho_{\text{KE}}(\mathbf{r}) = \mathbf{H}(\mathbf{r})\cdot d\mathbf{B}(\mathbf{r})$

d(potential energy density) = $d\rho_{\text{PE}}(\mathbf{r}) = \mathbf{E}(\mathbf{r})\cdot d\mathbf{D}(\mathbf{r})$

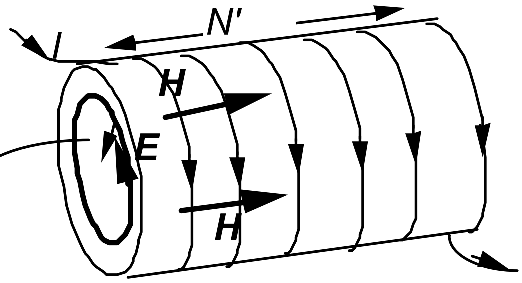
remark: momentum density $\rho_{\text{p}}(\mathbf{r}) = \mathbf{J}_{\text{E}}(\mathbf{r})/c^2$

in linear medium, $\mathbf{B} = \mu\mathbf{H}$, $\mathbf{D} = \varepsilon\mathbf{E} \Rightarrow u_M = \frac{1}{2}\mathbf{B}\cdot\mathbf{H}$, $u_E = \frac{1}{2}\mathbf{E}\cdot\mathbf{D}$

Example

ENERGY TRANSPORT IN A SOLENOID

$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H} = \mathbf{J}_{\mathrm{E}}$



In long solenoid $\mathbf{H}$ is axial,

$H = N'I,\ B = \mu H$

$\mathbf{E}$ is azimuthal, depends on radius ρ

Faraday, azimuthal path:

$$\oint d\mathbf{L}\cdot\mathbf{E} = -\frac{d}{dt}\int dA\cdot\mathbf{B} \Rightarrow 2\pi\rho E(\rho) = -\pi\rho^2 \mu N' \frac{dI}{dt} \quad ,$$

$$E(\rho) = -\frac{\rho\mu N'}{2}\frac{dI}{dt} \quad , \quad \mathbf{S} = \mathbf{E}\times\mathbf{H} = -\rho\frac{\mu N'^2}{2} I\frac{dI}{dt} \quad \hat{\rho}$$

Right-hand rule gives $\mathbf{H}$ as shown in figure for I as shown.

CASES

(a) I increasing, Lenz's law ⇒ $\mathbf{E}$ as shown ⇒

current flowing along E reduces field, **opposing increase**;

then $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is radial inwards, largest near outside,

carrying energy into the interior where the fields,

and thus the energy density, are increasing.

(b) I decreasing, Lenz's law ⇒ $\mathbf{E}$ opposite ⇒

current flowing along E increases field, **opposing decrease**;

then $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is radial outwards, largest near outside,

carrying energy out of the interior where the fields,

and thus the energy density, are decreasing.

Example: Inductance of a coaxial cable

In general, the magnetic energy is $E = \frac{1}{2} \int d^3\mathbf{r}\ \mathbf{B}(\mathbf{r})\bullet\mathbf{H}(\mathbf{r})$

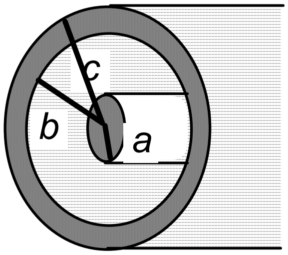
For an inductor, $E = \frac{1}{2} L I^2 \qquad \Rightarrow L = \frac{1}{I^2} \int d^3\mathbf{r}\ \mathbf{B}(\mathbf{r})\bullet\mathbf{H}(\mathbf{r})$

example:

Self-inductance of coaxial line

find inductance per unit length

$$\frac{dL}{d\,length} = \frac{2\pi}{\mu_o I^2} \int_0^c \rho\, d\rho \quad B(\rho)^2$$



for $\rho < a$, $J = \frac{I}{\pi a^2}$, Ampére: $2\pi\rho B = \mu_o \pi \rho^2 J$

$$\Rightarrow B = \frac{\mu_o I \rho}{2\pi a^2} \Rightarrow \frac{dL}{d\,length} = \frac{2\pi}{\mu_o I^2} \int_0^a \rho\, d\rho\, \left(\frac{\mu_o I \rho}{2\pi a^2} \right)^2 = \frac{\mu_o}{8\pi}$$

for $a < \rho < b$, $J = 0$, Ampére: $2\pi\rho B = \mu_o I$

$$\Rightarrow B = \frac{\mu_o I}{2\pi\rho} \quad \square \quad \frac{dL}{d\,length} = \frac{2\pi}{\mu_o I^2} \int_a^b \rho\, d\rho\, \left(\frac{\mu_o I}{2\pi\rho} \right)^2 = \frac{\mu_o}{2\pi} \ln \frac{b}{a}$$

for $b < \rho < c$, $J = -\frac{I}{\pi(c^2-b^2)}$, Ampere: $2\pi\rho B = \mu_o[I - \pi(\rho^2-b^2)J]$

$$\Rightarrow B = \frac{\mu_o I}{2\pi\rho} \frac{c^2-\rho^2}{c^2-b^2} \quad \Rightarrow \frac{dL}{d\,length} = \frac{2\pi}{\mu_o I^2} \int_b^c \rho\, d\rho\, \left(\frac{\mu_o I}{2\pi\rho} \frac{c^2-\rho^2}{c^2-b^2} \right)^2$$

$$= \frac{\mu_o}{2\pi} \frac{c^4 \ln \frac{c}{b} - 2c^2(c^2-b^2)/2 + (c^4-b^4)/4}{(c^2-b^2)^2}$$

Total inductance per unit length = sum of above 3 terms

4.5.B. MAGNETIC FORCE

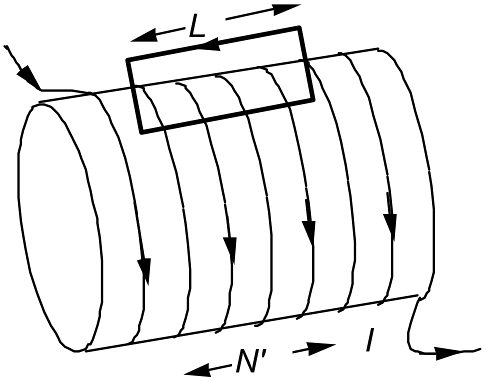
$\frac{\text{Force}}{\text{Area}}$ = magnetic pressure = energy density

: directed **out of** region of field

proof: $\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{\text{Energy/Displacement}}{\text{Area}}$

$= \frac{\text{Energy}}{\text{Displacement} \cdot \text{Area}} = \frac{\text{Energy}}{\text{Volume}}$ but oppositely directed

Example: SOLENOID WITH CORE



Ampere: $\oint \boldsymbol{H} \cdot d\boldsymbol{L} = I_{\text{free}}$

$(H_{\text{in}} - H_{\text{out}})_{||} L = N'IL$

$H_{\text{out}} \approx 0 \Rightarrow \quad H_{\text{in}} \approx N'I$

$B_{\text{in}} = \mu H_{\text{in}}$

$\approx \mu N'I > \mu_o H_{\text{in}}$

OUTWARD PRESSURE TO EXPAND SOLENOID

$$F_R = \partial E_{\text{magnetic}}/\partial R = \frac{\partial}{\partial R}(\pi R^2 L \times \boldsymbol{B} \cdot \boldsymbol{H})$$

$$= (2\pi RL) \times \left(\frac{1}{2} \boldsymbol{B} \cdot \boldsymbol{H}\right) \qquad = (\text{area}) \times (\text{pressure})$$

If winding is loose, pressure <u>on winding only</u> is $\frac{1}{2\mu_o} \boldsymbol{B}^2$

× energy density in region between winding and core.