Capstones in Physics: Electromagnetism

1. ELECTROSTATICS

(References to) "Introduction to Electrodynamics" by David J. Griffiths
3rd ed., Prentice-Hall, 1999. ISBN 0-13-805326-X

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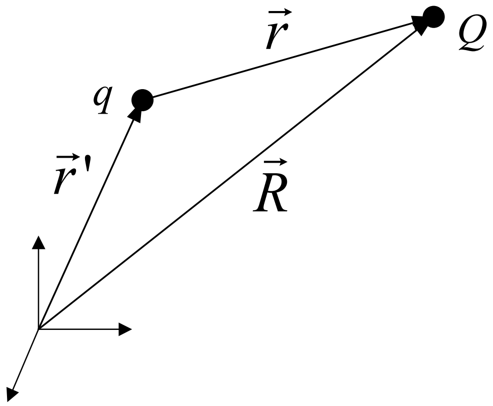
1.1 Electric Field

Coulomb's law

Force on a test charge *Q* due to a point charge *q*

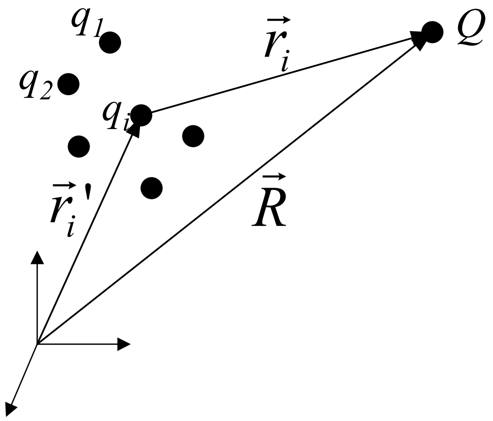


$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\widehat{r}$$

$$\vec{r} = \vec{R} - \vec{r}\,'$$

$$\varepsilon_0 = 8.85\times 10^{-12}\,\frac{C^2}{N\cdot m^2}$$

Permitivity of free space



$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\widehat{r}_1 + \frac{q_2}{r_2^2}\widehat{r}_2 + \cdots\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\widehat{r}_i$$

Electric field

$$\vec{F} = Q\vec{E} \quad \Rightarrow \quad \vec{E} \equiv \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\widehat{r}_i$$ superposition principle

For continuous charge distribution : $\vec{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r)}{r^2}\hat{r}dv$

Gauss's law

The flux through any surface enclosing charge Q_{enc} is $Q_{\text{enc}}/\varepsilon_0$.

For any closed surface, $\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

Divergence theorem: $\oint_S \vec{E}\cdot d\vec{a} = \int_V \left(\nabla\cdot\vec{E}\right) d\tau$

Q_{enc} in terms of the charge density ρ : $Q_{enc} = \int_V \rho\, d\tau$

$$\Rightarrow \quad \int_V \left(\nabla\cdot\vec{E}\right) d\tau = \int_V \left(\frac{\rho}{\varepsilon_0}\right) d\tau \quad \Rightarrow \quad \nabla\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho$$

The Curl of E

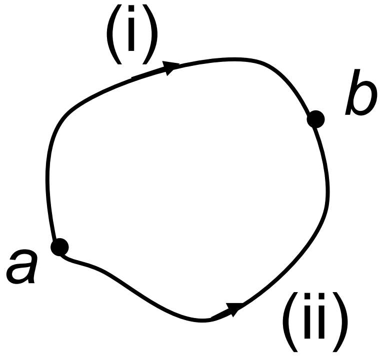
Integral of ***E*** around a closed path is zero:

$$\oint \vec{E}\cdot d\vec{l} = 0 \quad \Rightarrow \quad \nabla\times\vec{E} = 0$$

Stokes' theorem, $\oint \vec{E}\cdot d\vec{l} = \int_S \left(\nabla\times\vec{E}\right)\cdot d\vec{a}$

1.2 Electric potential

Because $\oint \vec{E}\cdot d\vec{l} = 0$, the line integral of ***E*** from ***a*** to ***b*** is the same for all paths.



Because the line integral is independent of path, we can define a function

$$V(\vec{r}) \equiv -\int_O^{\vec{r}} \vec{E} \cdot d\vec{l} \quad \text{(electric potential).}$$

O is some standard reference point, thus V depends only on the point $\boldsymbol{r}$.

The differential version: $\vec{E} = -\nabla V$

Poisson's equations and Laplace's equations

$$\begin{aligned} &\vec{E} = -\nabla V \\ &\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \Rightarrow \quad \nabla^2 V = -\frac{\rho}{\varepsilon_0} \quad \text{Poisson's equation} \\ &\nabla \times \vec{E} = 0 \end{aligned}$$

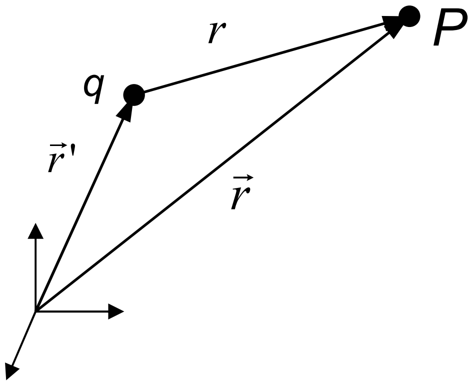
In regions $\rho=0$, $\nabla^2 V = 0$: Laplace's equation

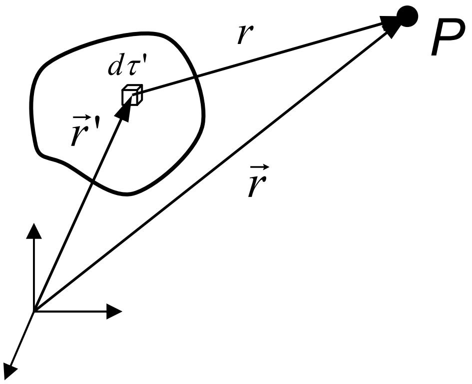
Potential of a localized charge distribution

The potential of a point charge q :

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$, where r is the distance from the charge to $\vec{r}$.

For a volume charge, $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho(\vec{r}\,')}{r} d\tau'$





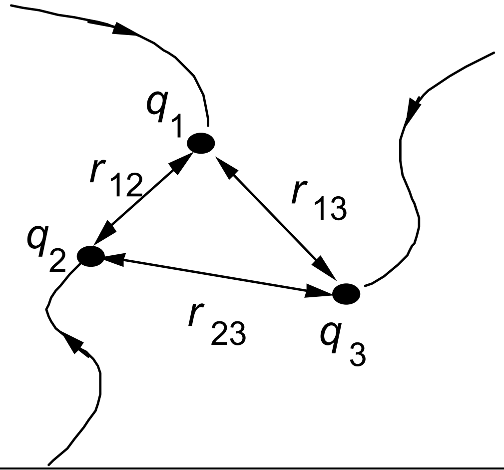
1.3 Work and energy in Electrostatics

ENERGY OF A CHARGE DISTRIBUTION

potential energy of distribution = energy required to assemble it

construct from pair potential energies $U_{ij} = \dfrac{q_i q_j}{4\pi\varepsilon_0 r_{ij}}$

step	energy
1. bring charge 1	0
2. bring charge 2	U_{12}
3. bring charge 3	$U_{13} + U_{23}$
4, 5,....	



total energy = $U_{12} + U_{13} + U_{23} + \ldots$

$$\boxed{U_{\text{total}} = \sum_{\text{pairs}} U_{ij}}$$

compare

$$U_{total} = \frac{1}{2}\sum_i \sum_{j\neq i} U_{ij} = \frac{1}{2}\sum_i \sum_{j\neq i} \frac{q_i q_j}{4\pi\varepsilon_0 r_{ij}} = \frac{1}{2}\sum_i q_i \sum_{j\neq i} \frac{q_j}{4\pi\varepsilon_0 r_{ij}} = \frac{1}{2}\sum_i q_i V_i$$

$\Rightarrow$ generalization to continuous distribution has factor $\frac{1}{2}$:

$$\boxed{U_{\text{total}} = \frac{1}{2} \int d^3\boldsymbol{r} \quad \rho(\boldsymbol{r})\, V(\boldsymbol{r}) = \frac{1}{2} \int d^3\boldsymbol{r} \int d^3\boldsymbol{R} \; \frac{\rho(\boldsymbol{r})\, \rho(\boldsymbol{R})}{4\pi\varepsilon_0 |\boldsymbol{r}-\boldsymbol{R}|}}$$

note this form doesn't work for idealized distributions,

including point charges: the energy integral diverges!

ENERGY OF AN ELECTRIC FIELD

potential energy of a charge distribution = energy required to assemble it

(note: can only compute if finite!) see "energy of a charge distribution"

counting argument → $U_{\text{total}} = \frac{1}{2}\int d^3\boldsymbol{r} \;\; \rho(\boldsymbol{r})\, V(\boldsymbol{r})$

express ρ in terms of electric field $\boldsymbol{E}(\boldsymbol{r})$: $\rho(\boldsymbol{r}) = \varepsilon_o \nabla\bullet\boldsymbol{E}(\boldsymbol{r})$

substitute: $U_{\text{total}} = \frac{\varepsilon_o}{2}\int d^3\boldsymbol{r} \;\; V(\boldsymbol{r})\, \nabla\bullet\boldsymbol{E}(\boldsymbol{r})$

vector identity: $\nabla\bullet(a\boldsymbol{A}) = (\nabla a)\bullet\boldsymbol{A} + a\,(\nabla\bullet\boldsymbol{A})$

substitute $a \to V$, $\boldsymbol{A} \to \boldsymbol{E}$:

$$U_{\text{total}} = \frac{\varepsilon_o}{2}\int d^3\boldsymbol{r} \;\; \{\nabla\bullet[V(\boldsymbol{r})\,\boldsymbol{E}(\boldsymbol{r})] - \boldsymbol{E}(\boldsymbol{r})\bullet\nabla V(\boldsymbol{r})]\}$$

express ∇V in terms of electric field $\boldsymbol{E}(\boldsymbol{r})$: $\boldsymbol{E}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r})$

divergence theorem: $\int_V d^3\boldsymbol{r}\, \nabla\bullet\boldsymbol{A}(\boldsymbol{r}) = \int_S d\boldsymbol{S}\bullet\boldsymbol{A}(\boldsymbol{r})$

substitute $\boldsymbol{A} \to V\boldsymbol{E}$, apply to expression for U_{total}:

$$U_{\text{total}} = \frac{\varepsilon_o}{2}\int dS \;\bullet[V(\boldsymbol{r})\,\boldsymbol{E}(\boldsymbol{r})] + \frac{\varepsilon_o}{2}\int d^3\boldsymbol{r} \;\; [\boldsymbol{E}(\boldsymbol{r})\bullet\boldsymbol{E}(\boldsymbol{r})]$$

first term → 0 when surface goes to ∞ :

area ~ R^2, but localized charges ⇒ $E \sim 1/R^2$, $V \sim 1/R$

conclude $\boldsymbol{U}_{\text{total}} = \int d^3\boldsymbol{r}\, \frac{\varepsilon_o}{2}\; \boldsymbol{E}(\boldsymbol{r})^2$

but surely $U_{\text{total}} = \int d\; d^3\boldsymbol{r}$ (energy density)

$$\boxed{\text{energy density} = \frac{1}{2}\,\varepsilon_o\, \boldsymbol{E}(\boldsymbol{r})^2}$$

1.4 Conductors

STATIC CONDUCTORS

Conductors have charges free to move.

<u>In steady state</u>, constant current ⇒ motion of charges
<u>In static state</u>, charges aren't moving ⇒ no forces ⇒

$$\boxed{\begin{array}{lll} \boldsymbol{E}(\boldsymbol{r}) = 0 \text{ everywhere inside static conductor} & & \\ \Rightarrow & \bullet \text{ charge density } \rho(\boldsymbol{r}) = 0 & \\ \text{and} & \bullet \text{ potential } V(r) = \text{constant} & \end{array}}$$

⇒ charge density only on surface, σ see "Surface charge"

Gauss' law ⇒ $\boldsymbol{E}_{\text{outside}} - \boldsymbol{E}_{\text{inside}} = \sigma/\varepsilon_o$ normal to surface

but $\boldsymbol{E}_{\text{inside}} = 0$, so

$$\boxed{\boldsymbol{E}_{\text{outside}} = \frac{\sigma}{\varepsilon_o}\,\hat{\boldsymbol{n}} \qquad \text{at surface of conductor}}$$

<u>Pressure on a conductor due to electric field</u>

consider surface charge on area dS $\quad Q = \sigma\, dS$

force due to other charges $\quad \boldsymbol{F} = Q\,\boldsymbol{E}_{\text{external}}$

But $E_{\text{external}} = E_{\text{due to surface charge}}$ ⇒ $\quad E_{\text{external}} = \frac{1}{2}\,E$

pressure $p = F_{\text{normal}}/dS$ ⇒ $\quad p = \sigma\, E_{\text{ext}} = \frac{1}{2}\,\sigma E$

$$\boxed{\text{pressure} \qquad p = \frac{\sigma^2}{2\varepsilon_o} = \frac{\varepsilon_o}{2}\,E^2 \quad \text{on surface of conductor}}$$

alternately, $F = -\dfrac{\delta E_{\text{potential}}}{\delta \mathit{position}}$ see "Electrostatic Forces"

$E_{\text{pot}} = \frac{\varepsilon}{2}\,E^2 \times vol$, move surface ⇒ $\delta vol = S \times \delta \mathit{position}$ √

CAPACITANCE

recall a conductor is an equipotential,

<u>claim:</u> potential V on an isolated conductor ~ its charge Q

proof: $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ is a linear relation, determines V

<u>definition</u>: capacitance $C \equiv Q/V$ for isolated conductor

2 conductors with equal, opposite charges are a <u>capacitor</u> .

<u>definition:</u> capacitance $C \equiv \dfrac{Q}{|V_2 - V_1|} = \dfrac{Q}{\Delta V}$ for pair of conductors

energy stored by charging = $\int_0^Q dQ \;\; V(Q) = \int_0^Q dQ \;\; Q/C$

$$\boxed{\text{energy stored} = \frac{Q^2}{2C} = \frac{1}{2}\, Q\, V = \frac{1}{2}\, C\, V^2}$$

<u>example:</u> 2 large flat plates

area S, dielectric thickness d

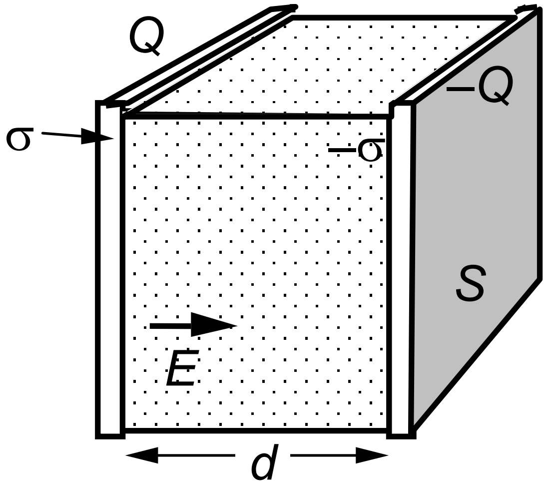
assume charges ±Q on plates

this is free charge!

$\Rightarrow \sigma = Q/S \Rightarrow$

$E - E_{\text{cond}} = \sigma/\varepsilon_0$

$E_{\text{cond}} = 0 \;\Rightarrow E = \sigma/\varepsilon_0$



$$\Delta V = -\int d\mathbf{L} \cdot \mathbf{E} = d\,E = \frac{d\sigma}{\varepsilon_0} \qquad \Rightarrow C = \frac{Q}{\Delta V} = \frac{\sigma S}{d\,\sigma/\varepsilon_0} = \frac{\varepsilon_0 S}{d}$$

MIRROR IMAGES

Uniqueness theorem: The solution to Laplace's equation in some volume is uniquely determined if V is specified on the boundary surface S.

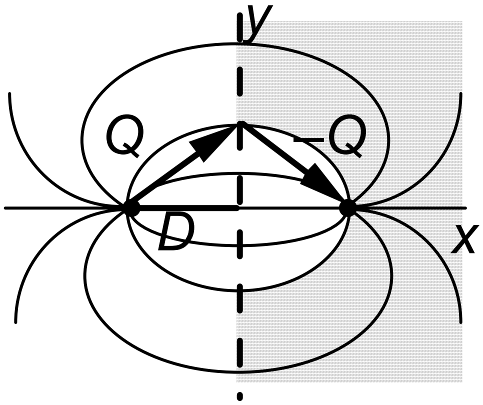
At a plane interface, the potential can be described by a trick:

simplest example:

point charge Q at $(-D,0,0)$

conducting plane $x = 0$

need to find induced charge to make $\boldsymbol{E} \perp$ to plane



Potential for x<0 (V=0 for x>0)

$$V(x,y,z) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{(x+D)^2+y^2+z^2}} - \frac{1}{\sqrt{(x-D)^2+y^2+z^2}}\right]$$

Field in midplane

$$E_x(x=0,y,z) = 2\,\frac{Q}{4\pi\varepsilon_0(D^2+y^2+z^2)}\;\frac{D}{\sqrt{D^2+y^2+z^2}}$$

surface charge density $\sigma(y,z) = \varepsilon_0 E\bullet\hat{\boldsymbol{n}} = \dfrac{-QD}{2\pi(D^2+y^2+z^2)^{3/2}}$

Field on right = 0, field on left = $\boldsymbol{E}_Q + E_{\text{induced}}$

$$\boldsymbol{E}_{\text{left}} = \frac{Q(\boldsymbol{R}+D\hat{\boldsymbol{x}})}{4\pi\varepsilon_0[(x+D)^2+y^2+z^2]^{3/2}} + \frac{-Q(\boldsymbol{R}-D\hat{\boldsymbol{x}})}{4\pi\varepsilon_0[(x-D)^2+y^2+z^2]^{3/2}}$$

Note total induced charge = $-Q$ (proof: Gauss' law applied to surface enclosing all charges, field ~ $1/r^3$)

generally, superposition ⇒ mirror-image charge distribution

More complicated: image in surface of dielectric insulator

<u>Other geometry:</u> can also find images for surfaces:

- spherical surface
- cylindrical surface