Capstones in Physics: Electromagnetism

2. ELECTRIC FIELDS IN MATTER

2.1. Multipole Expansion

A. Multipole expansion of potential
B. Dipole moment
C. Electric field of dipole
D. Dipole in an electric field

2.2. Charges in Materials

A. Polarization
B. Displacement field
C. Dielectric boundaries

(References to) "Introduction to Electrodynamics" by David J. Griffiths
3rd ed., Prentice-Hall, 1999. ISBN 0-13-805326-X

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2.1 MULTIPOLE EXPANSION

2.1.A. MULTIPOLE EXAPANSION OF POTENTIAL

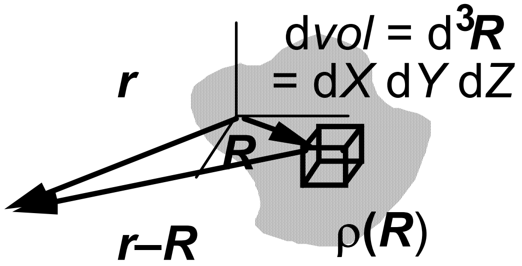
start from

$$V(\boldsymbol{r}) = \int d^3\boldsymbol{R}\ \frac{\rho(\boldsymbol{R})}{4\pi\varepsilon_o|\boldsymbol{r}-\boldsymbol{R}|}$$

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{R}|} = [(\boldsymbol{r}-\boldsymbol{R})^2]^{-1/2}$$

$$= [r^2 - 2\boldsymbol{r}\bullet\boldsymbol{R} + R^2]^{-1/2}$$

$$= \frac{1}{r}\left[1 - 2\frac{\boldsymbol{r}\bullet\boldsymbol{R}}{r^2} + \left(\frac{R}{r}\right)^2\right]^{-1/2}$$



<u>expand for $r \gg R$:</u> Taylor series in $\frac{R}{r} \ll 1$

to first order: $$\frac{1}{|\boldsymbol{r}-\boldsymbol{R}|} \approx \frac{1}{r}\left[1 + \frac{\boldsymbol{r}\bullet\boldsymbol{R}}{r^2} + O\left(\frac{R}{r}\right)^2\right]$$

each term of the integrand factors

into a product of a function of $\boldsymbol{r}$ times a function of $\boldsymbol{R}$:

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_o r}\int d^3\boldsymbol{R}\ \rho(\boldsymbol{R}) \quad + \quad \frac{\hat{\boldsymbol{r}}}{4\pi\varepsilon_o r^2}\bullet\int d^3\boldsymbol{R}\ \rho(\boldsymbol{R})\ \boldsymbol{R}$$

total charge Q dipole moment $\boldsymbol{p}$

to all orders: let $\alpha = \boldsymbol{r}\angle\boldsymbol{R}$, the angle between $\boldsymbol{r}$ and $\boldsymbol{R}$. Then $\cos\alpha = \hat{\boldsymbol{r}} \bullet \hat{\boldsymbol{R}}$,

and $$\frac{1}{|\boldsymbol{r}-\boldsymbol{R}|} = \frac{1}{r}\left[1 - 2\frac{R}{r}\cos\alpha + \left(\frac{R}{r}\right)^2\right]^{-1/2} = \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{R}{r}\right)^l P_l(\cos\alpha)$$

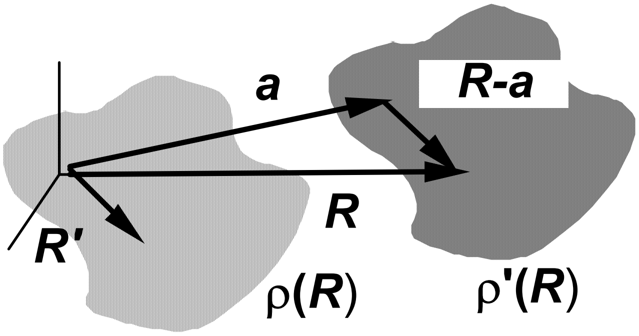
where $P_l(\cos\alpha) = \lim_{u\to 0}\frac{1}{l!}\frac{\partial^l}{\partial u^l}[1 - 2u\cos\alpha + u^2]^{-1/2}$

is a <u>Legendre Polynomial of order l</u>

2.1.B. DIPOLE MOMENT $\boldsymbol{p} = \int d^3\boldsymbol{R}\ \rho(\boldsymbol{R})\,\boldsymbol{R}$

consider another charge distribution

$\rho'(\boldsymbol{R}) = \rho(\boldsymbol{R} - \boldsymbol{a})$



Then its dipole moment is

$$\boldsymbol{p'} = \int d^3\boldsymbol{R}\ \rho'(\boldsymbol{R})\,\boldsymbol{R} = \int d^3\boldsymbol{R}\ \rho(\boldsymbol{R} - \boldsymbol{a})\,\boldsymbol{R}$$

change variables: $\boldsymbol{R'} = \boldsymbol{R} - \boldsymbol{a}$, $\boldsymbol{R} = \boldsymbol{R'} + \boldsymbol{a}$, $d\boldsymbol{R} = d\boldsymbol{R'}$

$$\boldsymbol{p'} = \int d^3\boldsymbol{R'}\ \rho(\boldsymbol{R'})\,(\boldsymbol{R'} + \boldsymbol{a}) = \int d^3\boldsymbol{R'}\ \rho(\boldsymbol{R'})\,\boldsymbol{R'} + \boldsymbol{a}\int d^3\boldsymbol{R'}\ \rho(\boldsymbol{R'})$$

$$\boxed{\boldsymbol{p'} = \boldsymbol{p} + \boldsymbol{a}\,Q_{\text{total}}}$$

this is similar to the rule for the moment of inertia, $I' = I + \boldsymbol{a}\,M_{\text{total}}$

2.1.C. ELECTRIC FIELD OF DIPOLE

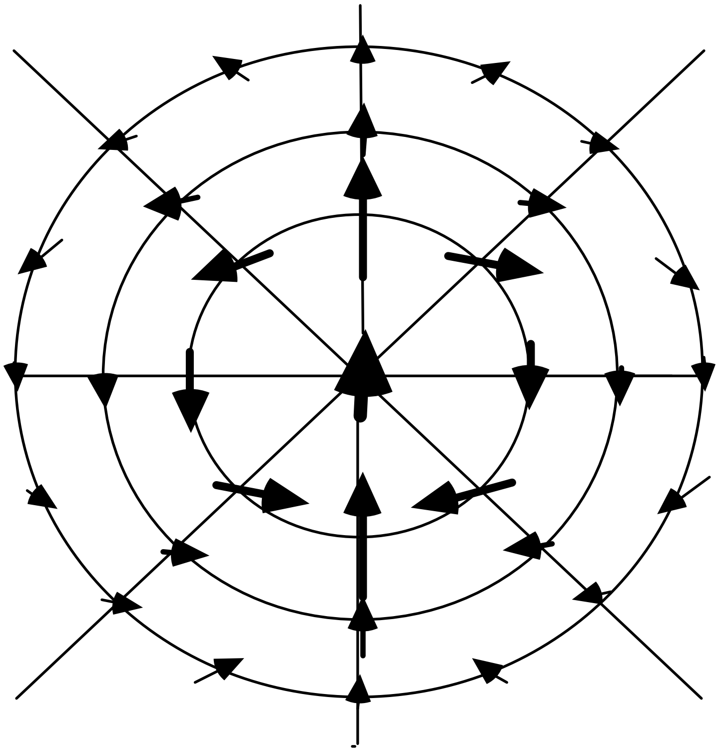
Far from the distribution

$\boldsymbol{E}(\boldsymbol{r}) \rightarrow$

$$\frac{q_{\text{total}}\hat{\boldsymbol{r}}}{4\pi\varepsilon_0 r^2} \quad \text{monopole}$$

$$+\ \frac{3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{4\pi\varepsilon_0 r^3} \quad \text{dipole}$$

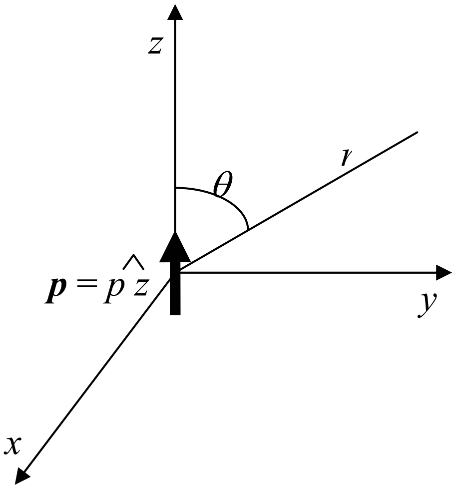
+ ... higher multipoles



If ***p*** lies at the origin and points in the *z* direction, the potential is

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}\cdot\mathbf{p}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\,.$$

The electric field of the dipole is $\mathbf{E} = -\nabla V$:



$$E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{\partial V}{r\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0$$

Thus

$$\mathbf{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}).$$

Using $\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$,

$$\mathbf{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}(3\cos\theta\,\hat{r} - \hat{z})$$

$$= \frac{1}{4\pi\varepsilon_0 r^3}(3p\cos\theta\,\hat{r} - p\,\hat{z})$$

$$= \frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi\varepsilon_0 r^3}$$

2.1.D. DIPOLE IN AN ELECTRIC FIELD

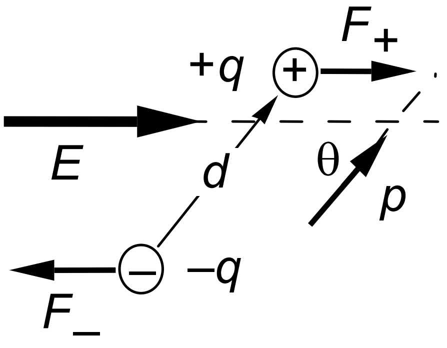
dipole moment

$\mathbf{p} = q\mathbf{d}$

in uniform field $\mathbf{E}$

picture plane of $\mathbf{E}$, $\mathbf{p}$

NO NET FORCE



total force $= \mathbf{F}_+ + \mathbf{F}_-$

both directed along $\mathbf{E}$ ⇒

total force $= q\mathbf{E} - q\mathbf{E} = 0$

Instead of a net force, there is a <u>torque</u> on the dipole

$$\tau = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

let $\mathbf{r}$ = vector distance from center of dipole, $\mathbf{r}_\pm = \pm\mathbf{d}/2$, then

$$\tau = \frac{\mathbf{d}}{2} \times (qE) - \frac{\mathbf{d}}{2} \times (-qE) = q(d \times \mathbf{E}) = \mathbf{p} \times \mathbf{E}$$

torque on dipole $= \tau = \mathbf{p} \times \mathbf{E}$,
direction ⊥ to both $\mathbf{E}$ and $\mathbf{p}$, magnitude $= Ep \sin\theta_{Ep}$

the torque tends to make the dipole line up with the field

<u>work done by field to align dipole</u> $\mathrm{d}W = \tau\, \mathrm{d}\theta = pE \sin\theta\, \mathrm{d}\theta$

start from $\mathbf{p} \perp \mathbf{E}$ ⇒ energy $= \int_{\pi/2}^{\theta} \mathrm{d}\theta\; pE\sin\theta = -pE\cos\theta$

Alignment energy $U = -\mathbf{p}\bullet\mathbf{E}$, minimum energy when $\mathbf{p} \parallel \mathbf{E}$

In non-uniform field $\mathbf{E}$

$$\mathbf{F} = -\nabla U = \nabla(\mathbf{p}\cdot\mathbf{E}) = \mathbf{p}\times(\nabla\times\mathbf{E}) + (\mathbf{p}\cdot\nabla)\mathbf{E} = (\mathbf{p}\cdot\nabla)\mathbf{E}$$

2.2. CHARGES IN MATERIALS

Materials are made up of atoms and/or molecules.

Atoms and molecules are made up of electrons and nuclei.

The nuclei have: more than 99.9% of the mass
positive charges $+Ze$ (Z = atomic number)

The electrons have: negative charges $-e$
most of the material's ability to respond to fields

Charges divide into <u>free charges</u> and <u>bound charges</u>

Conductors have free charges, **insulators** don't.

Most solid conductors are crystalline (often microcrystals).

Solid insulators: may be crystalline, or polymers (long molecules).

Fluids are insulators unless they contain charged **ions**.

All materials, even the vacuum, are **dielectrics**:
they have bound charges which move in electric fields.

2.2.A. POLARIZATION

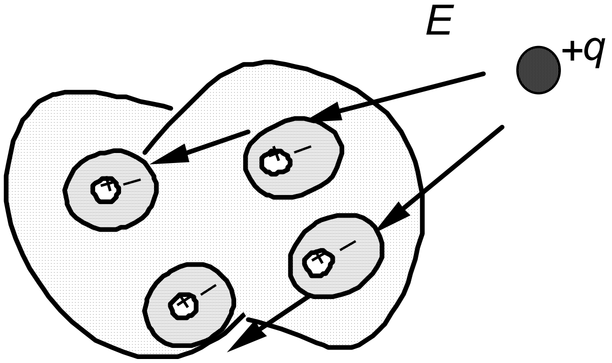
A <u>dielectric</u> is an insulator

⇒ charges can't move far

But they <u>can</u> move a little way

in response to a field

see "Oscillator Model"

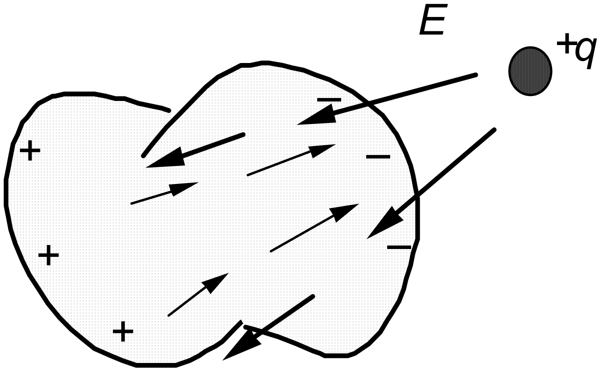


the charges accumulate

on the surface ⇒

the induced charges

cause an attractive force!



the induced charges cause an induced field

which <u>reduces</u> the internal field

> in a <u>linear dielectric</u>, $\mathbf{E}_{\text{internal}} = \frac{1}{\kappa}\ \mathbf{E}_{\text{applied}}$

handy rule of thumb: $\varepsilon_0 \rightarrow \kappa\,\varepsilon_0$

The polarization affects the potential energy

OSCILLATOR MODEL OF DIELECTRIC POLARIZATION

static charges $\pm q$

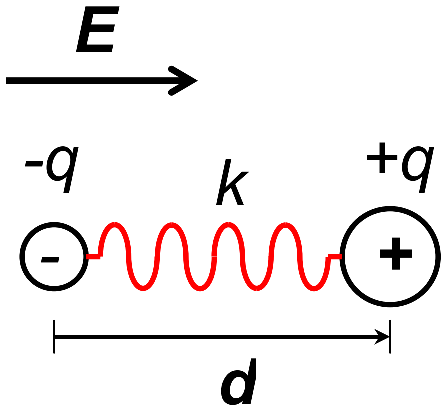
separation $= \mathbf{d}$

dipole moment $= q\mathbf{d}$

reduced mass m, spring constant k

vibration frequency $\Omega_0 = \sqrt{k/m}$

electric field $= \mathbf{E}$



<u>balance of forces on negative charge:</u>

$$\text{electric force} = -q\mathbf{E} = -\text{ restoring force} = -k\,\mathbf{d}$$

$\Rightarrow$ DIPOLE MOMENT $\mathbf{p} = q\mathbf{d} = q\,(\,q\mathbf{E}/k\,)$

$\Rightarrow$ <u>polarization density</u> $\mathbf{P} = n\mathbf{p} = \chi(\text{static})\varepsilon_0\mathbf{E}$

where n = density of oscillators per unit volume (Griffiths: N)

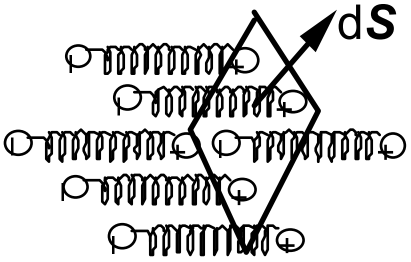
$$\chi(\text{static}) = \text{electric susceptibility} = \frac{nq^2}{\varepsilon_0 k} = \frac{nq^2}{\varepsilon_0 m\Omega_0^2}$$

NOTE: the description has to be modified

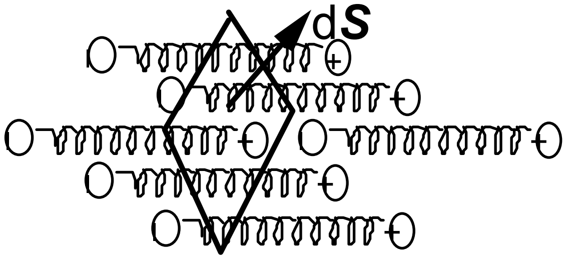
when $\mathbf{E}$ changes on a time scale of $1/\Omega_0$

BOUND CHARGES

consider a surface d$\boldsymbol{S}$ in a dielectric (Griffiths: d$\boldsymbol{a}$)



apply electric field $\boldsymbol{E}$



charges move distance $\boldsymbol{d}-\boldsymbol{d}_O$ see "Oscillator Model"

$\Rightarrow$ charges within $\boldsymbol{d}-\boldsymbol{d}_O$ cross surface

$\Rightarrow$ volume of charges crossing surface = $(\boldsymbol{d}-\boldsymbol{d}_O)\cdot d\boldsymbol{S}$

n = density of bound charges per unit volume (Griffiths: N)

$\Rightarrow$ charge moving through surface =

$$dq = n\,q\,(\boldsymbol{d}-\boldsymbol{d}_O)\cdot d\boldsymbol{S} = \boldsymbol{P}\cdot d\boldsymbol{S}$$

<u>conclusions:</u>

1. <u>at surface of dielectric</u>, charges accumulate see "Surface Conditions"

   $\Rightarrow$ surface charge density $\sigma_{\text{bound}} = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = P_\perp$

   where $\hat{\boldsymbol{n}}$ = unit normal to surface

2. <u>inside dielectric</u>, $-\nabla\cdot\boldsymbol{P}(\boldsymbol{r}) = \rho_{\text{bound}} = nq$

   proof: consider any closed volume, then
   charge moving out of volume = $\oint d\boldsymbol{S}\cdot\boldsymbol{P} = \int d\mathit{vol}\,\nabla\cdot\boldsymbol{P}$

   but conservation of charge $\Rightarrow$

   charge moving out of volume = – charge in volume

   $$= -\int d\mathit{vol}\;\rho_{\text{bound}} = \int d\mathit{vol}\,\nabla\cdot\boldsymbol{P}$$

   true for <u>any</u> volume $\Rightarrow$ integrands equal √

2.2.B. DISPLACEMENT FIELD

Aim: to find electric field $\boldsymbol{E}$ in terms of free charges ρ_{free}

Problem: $\boldsymbol{E}$ comes from all charges, bound and free

$$\varepsilon_0 \nabla\bullet\boldsymbol{E}(\boldsymbol{r}) = \rho(\boldsymbol{r}) \qquad \text{see "Gauss' Law"}$$

Solution: exploit relation between bound charges and $\boldsymbol{P}$,

$$\nabla\bullet\boldsymbol{P}(\boldsymbol{r}) = -\,\rho_{\text{bound}} \qquad \text{see "Surface Polarization"}$$

add to Gauss' law: $\nabla\bullet(\varepsilon_0\boldsymbol{E} + \boldsymbol{P}) = \rho - \rho_{\text{bound}} = \rho_{\text{free}}$

$$\boxed{\text{define displacement field} \quad \boldsymbol{D} \equiv \varepsilon_0\boldsymbol{E} + \boldsymbol{P} \;\Rightarrow \qquad \nabla\bullet\boldsymbol{D} = \rho_{\text{free}}}$$

once we know $\boldsymbol{D}$, can find $\boldsymbol{E}$ from relation between $\boldsymbol{E}$ and polarization $\boldsymbol{P}$,

$$\boldsymbol{P} = n\boldsymbol{p} = \boldsymbol{P}_0 + \chi(\text{static})\varepsilon_0\boldsymbol{E} \qquad \text{see "Oscillator Model"}$$

substitute, setting $\boldsymbol{P}_0 = 0$ as is almost always true

$$\boldsymbol{D} = \varepsilon_0\boldsymbol{E} + \boldsymbol{P} = \varepsilon_0\boldsymbol{E} + \chi(\text{static})\varepsilon_0\boldsymbol{E}$$

$$\boxed{\boldsymbol{D} = \varepsilon\boldsymbol{E} = \kappa\varepsilon_0\boldsymbol{E} \qquad \text{where} \qquad \varepsilon = \kappa\varepsilon_0 \;\text{ and }\; \kappa = 1 + \chi(\text{static})}$$

ε = permittivity, χ = polarizability, κ = dielectric constant

so the method is:

1. Find $\boldsymbol{D}$ from free charges
2. Find $\boldsymbol{E}$ from $\boldsymbol{D}$

see "Surface Conditions"

see "Method of Images"

see "Capacitors"

Griffiths notation $\kappa \to K$, $\chi(\text{static}) \to \chi_{\text{electric}}$

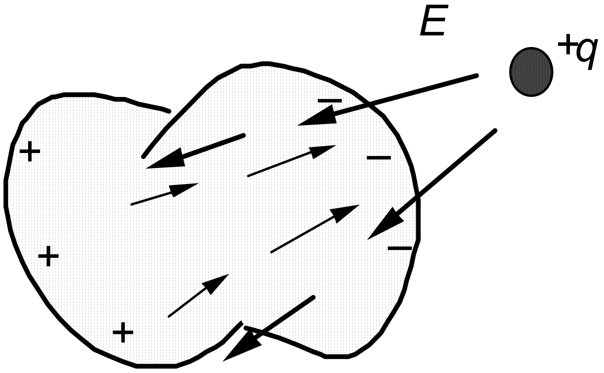
2.2.C. DIELECTRIC BOUNDARIES

the polarization charge
accumulates at the surface
of a dielectric ⇒

surface charge density

$\sigma_{\text{bound}} = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} = P_\perp$

where $\hat{\boldsymbol{n}}$ = unit normal
to the surface



see "Surface Polarization"

This implies that the electric field is discontinuous:

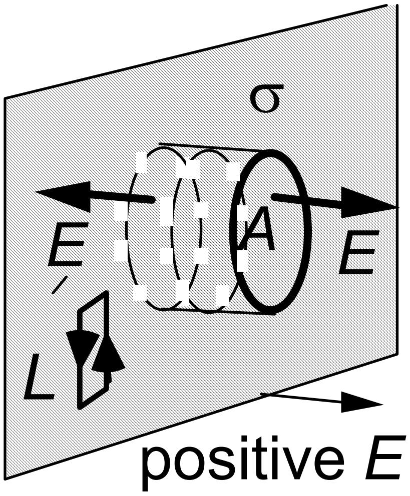
recall flat sheet of charge

choose "Gaussian pillbox"

$$\oint \boldsymbol{E}\cdot d\boldsymbol{A} = \frac{q_{\text{enclosed}}}{\varepsilon_0}$$

$$-E_{\text{left}}A + E_{\text{right}}A = \frac{\sigma A}{\varepsilon_0}$$

$E_{\text{right}} - E_{\text{left}} = \dfrac{\sigma}{\varepsilon_0}$ ⊥ to surface



Apply same reasoning to the integral form of $\nabla\cdot\boldsymbol{D} = \rho_{\text{free}}$

⇒ Perpendicular component of **D** is continous:

$\boldsymbol{D}_\perp \equiv \boldsymbol{D}\cdot\hat{\boldsymbol{n}}$ = same on both sides, $\boldsymbol{E}_{||}$ = same on both sides

alternate argument for $\boldsymbol{E}_{||}$:

apply $\oint \boldsymbol{E}\cdot d\boldsymbol{L} = 0$ to path near surface ⇒ $\boldsymbol{L}\cdot(\boldsymbol{E}_{\text{left}} - \boldsymbol{E}_{\text{right}}) = 0$

DIELECTRIC SPHERE

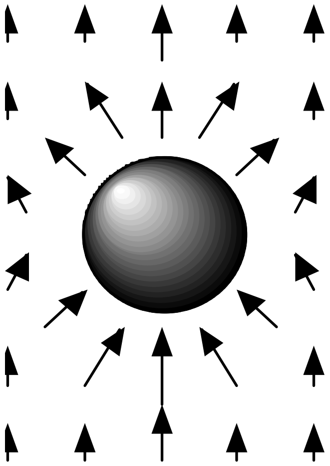
dielectric sphere, charge 0

radius R, center at $\boldsymbol{r} = 0$

external electric field $\boldsymbol{E} = E\,\hat{\boldsymbol{z}}$

$\Rightarrow$ potential $V(\boldsymbol{r}) \to V_o - E\,z$

far from the sphere



Axially-symmetric solutions to Laplace's equation, $\nabla^2 V = 0$:

$$V(r, \theta) = \sum_{n=0}^{\infty} \left(A_n\, r^n + B_n / r^{n+1}\right)\ P_n(\cos\theta) \text{ outside}$$

and $$V(r, \theta) = \sum_{n=0}^{\infty} \left(C_n\, r^n + D_n / r^{n+1}\right)\ P_n(\cos\theta) \text{ inside}$$

(*i.e.* different coefficients)

To get the coefficients, use orthogonality condition,

$$\int_{-1}^{1} d(\cos\theta)\ P_m(\cos\theta)\, P_n(\cos\theta) = \frac{2}{2n+1}\ \delta_{mn} \qquad \text{for fixed } r.$$

•Condition at $\boldsymbol{r} \to 0$

Inside the sphere the potential is finite $\Rightarrow$ $D_n = 0$ for all n.

•Condition at $\boldsymbol{r} \to \infty$

Outside sphere $V(\boldsymbol{r}\to\infty) \to -E\,z + V_o$ where V_o = constant.

$P_0 = 1$,$P_1 = \cos\theta$ $\Rightarrow V(\boldsymbol{r}\to\infty) = V_o\, P_0(\cos\theta) - E\, r\, P_1(\cos\theta)$

$\Rightarrow A_0 = V_0$, $A_1 = -E$; All the other A_n have to be zero.

The B_n are not constrained by these boundary conditions, but we can find them from the continuity conditions.

continued

DIELECTRIC SPHERE *continued*

•Continuity of *V* at interface

Coefficients inside, outside related by continuity conditions at surface $r = R$: V same just inside and just outside,

$$V(R,\theta) = \sum_{n=0}^{\infty} C_n R^n P_n(\cos\theta) = \sum_{n=0}^{\infty} \left(A_n R^n + \frac{B_n}{R^{n+1}}\right) P_n(\cos\theta)$$

$$V(R,\theta)_{\text{in}} - V(R,\theta)_{\text{out}} = 0 = \sum_{n=0}^{\infty} \left[(C_n R^n) - \left(A_n R^n + \frac{B_n}{R^{n+1}}\right)\right] P_n(\cos\theta)$$

multiply by $P_m(\cos\theta)\, d(\cos\theta)$ and integrate ⇒

coefficient of each $P_n(\cos\theta) = 0$ ⇒ $C_n = A_n + \dfrac{B_n}{R^{2n+1}}$

•Continuity of $D_\perp$ at interface

Similarly, the perpendicular *i.e.* radial component of $\boldsymbol{D} = \varepsilon \boldsymbol{E} = -\varepsilon_o \varepsilon_r \nabla V$

has to be continuous at the surface,

giving $D_{\perp\text{out}} - D_{\perp\text{in}} = \varepsilon_o \dfrac{\partial}{\partial r}\left(\varepsilon_r V(r,\theta)_{in} - V(r,\theta)_{out}\right)_{r=R} = 0$.

multipole expansions →

$$0 = \sum_{n=0}^{\infty} \frac{\partial}{\partial r}\left[\varepsilon_r (C_n r^n) - (A_n r^n + B_n / r^{n+1})\right]_{r=R} P_n(\cos\theta)$$

coeff. of each P_n = zero ⇒ $\varepsilon_r C_n = A_n - \dfrac{n+1}{n} \dfrac{B_n}{R^{2n+1}}$ for $n > 0$

Solve together with equation for continuity of V ⇒

$$(\varepsilon_r - 1) C_n = -\frac{2n+1}{n} \frac{B_n}{R^{2n+1}} , \qquad \frac{B_n}{R^{2n+1}} = (1 - \varepsilon_r) \frac{n}{2n+1} C_n ,$$

$$C_n = A_n + (1 - \varepsilon_r) \frac{n}{2n+1} C_n = A_n \left(1 - \frac{n}{2n+1}(\varepsilon_r - 1)\right)^{-1} ,$$

so both B_n and C_n are proportional to A_n for $n > 0$.

continued

DIELECTRIC SPHERE *continued*

for $n = 0$ $A_0 = C_0 = V_0$ = uniform constant voltage

Continuity of $\boldsymbol{D}_{\text{perp}} \Rightarrow \frac{\partial}{\partial r}\left[\varepsilon_r\left(C_0 r^0\right) - \left(A_0 r^0 + \frac{B_0}{r^{0+1}}\right)\right]_{r=R} = \frac{B_0}{R^2} = 0$

$\Rightarrow D_0 = B_0 = 0$, $A_0 = C_0 = V_0$ = uniform constant voltage

for $n = 1$ $A_1 = -E, \quad D_1 = 0$

$$C_1 = A_1 \frac{3}{\varepsilon_r + 2} = -E\frac{3}{\varepsilon_r + 2}, \quad B_1 = \frac{1-\varepsilon_r}{3} R^3 C_1 = \frac{\varepsilon_r - 1}{2 + \varepsilon_r} R^3 E,$$

for $n = 2, 3, \ldots$ $A_n = B_n = C_n = D_n = 0$

Consequently,

$$V(r,\theta) = \begin{cases} V_0 - \dfrac{3E}{\varepsilon_r + 2} r\cos\theta & \textbf{inside} \\ V_0 - E\left(r - \dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\dfrac{R^3}{r^2}\right)\cos\theta & \textbf{outside} \end{cases}$$

and

$$\mathbf{E} = \begin{cases} \dfrac{3E}{\varepsilon_r + 2}\hat{z} & \textbf{inside} \\ E\,\hat{z} + \left(\dfrac{\varepsilon_r - 1}{\varepsilon_r + 2}\dfrac{R^3}{r^3}\right)(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) & \textbf{outside} \end{cases}$$

Polarization

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\mathbf{E}_{in} = \varepsilon_0(\varepsilon_r - 1)\frac{3E}{\varepsilon_r + 2}\hat{z} = \frac{3\varepsilon_0(\varepsilon_r - 1)}{\varepsilon_r + 2}E\,\hat{z}$$

Induced electric potential when $r \to \infty$

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}\cdot\mathbf{p}}{r^2}$$, where $\mathbf{p} = \left(\frac{4\pi}{3}R^3\right)\mathbf{P}$

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi}{3}R^3\right)3\varepsilon_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)\frac{E\cos\theta}{r^2}$$

$$= \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)\frac{R^3}{r^2}E\cos\theta$$

Induced surface charge density

$$\sigma = \mathbf{P}\cdot\hat{r} = 3\varepsilon_0\left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2}\right)E\cos\theta$$